Markov Chain Example (Exercise #6.3 from Book)

An offspring of a black dog is black with probability 0.6 and brown with probability 0.4.
An offspring of a brown dog is black with probability 0.2 and brown with probability 0.8.

a) Write the transition probability matrix of this Markov chain.
b) Rex is a brown dog. Compute the probability that his grandpuppy is black.

Define 1 = "black" 2 = "brown"

X_t = color of dog State space = $Im(X_t) = \{1, 2\}$ (Black, Brown)

(a) Transition probability matrix : P_{ij} where i = current j = future

$P_{11} = 0.6$ ← P(Black puppy | Black dog)
$P_{12} = 0.4$ ← P(Brown puppy | Black dog)
$P_{21} = 0.2$ ← P(Black puppy | Brown dog)
$P_{22} = 0.8$ ← P(Brown puppy | Brown dog)

Now we can make the (1-step) transition matrix

$$P = \begin{matrix} \\ 1 \\ 2 \end{matrix} \begin{matrix} \begin{matrix} 1 & \quad 2 \end{matrix} \\ \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \end{matrix} = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix}$$

(i = row, j = column)

(b) Rex is brown dog. We want probability that his grandpuppy is black.
So we want to predict 2-steps ahead.

State space = $\{1, 2\}$
Initial Distribution : $P_0 = [0 \quad 1]$ (Black, Brown)
2-step Transition matrix. $P^{(2)} = P \cdot P$

$$P^{(2)} = P \cdot P = \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} \begin{bmatrix} 0.6 & 0.4 \\ 0.2 & 0.8 \end{bmatrix} = \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix}$$

Prediction

$$P_2 = P_0 \cdot P^{(2)} = P_0 \cdot P \cdot P$$

$$= [0 \quad 1] \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} = [0.28 \quad 0.72]$$

(Black, Brown)

P(Black grand puppy | Brown dog) = 0.28

Suppose I'm going to adopt a dog. There's a 60% probability that this dog is black & 40% probability this dog is brown.

Recall: State space = $\{1, 2\}$ (1 = black, 2 = brown)

Initial Distribution: $P_0 = \begin{bmatrix} 0.6 & 0.4 \end{bmatrix}$ (black, brown)

What is probability that this dog's grandpuppy is black? 37.6% or 0.376

$$P_2 = P_0 \cdot P^{(2)}$$

$$= P_0 \cdot P \cdot P$$

$$= \begin{bmatrix} 0.6 & 0.4 \end{bmatrix} \begin{bmatrix} 0.44 & 0.56 \\ 0.28 & 0.72 \end{bmatrix} = \begin{bmatrix} 0.376 & 0.624 \end{bmatrix}$$

(black) (brown)