Let $X_1, X_2, \dots, X_n$ be *iid* random variables with pdf:

$$f_X(x) = \begin{cases} (\theta+1)x^{\theta} & \text{for } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

We observe data values: $x_1 = 0.84,\ x_2 = 0.51,\ x_3 = 0.95$

1. Find the method of moments (MoM) estimate for θ
2. Find the maximum likelihood (MLE) estimate for θ

Answer:

1. Methods of Moments
   Since there's only one unknown parameter, we will just use the 1^{st} moment. The 1^{st} (population) moment is $\mu_1 = E(X)$

$$\mu_1 = E(X) = \int_{-\infty}^{\infty} xf(x)dx = \int_0^1 x\cdot(\theta+1)x^{\theta}dx = (\theta+1)\int_0^1 x^{\theta+1}dx = \frac{\theta+1}{\theta+2}x^{\theta+2}|_{x=0}^{1} = \frac{\theta+1}{\theta+2}$$

   The 1^{st} sample moment is $m_1 = \frac{\sum_{i=1}^n x_i}{n} = \bar{x}$

   Set the 1^{st} (population) moment equal to 1^{st} sample moment, and solve for θ.

$$\begin{aligned} \mu_1 = E(X) &\overset{set}{=} \bar{X} = m_1 \\ \implies & \frac{\theta+1}{\theta+2} = \bar{x} \\ \implies & \theta + 1 = \theta\bar{x} + 2\bar{x} \\ \implies & \theta - \theta\bar{x} = 2\bar{x} - 1 \\ \implies & \theta(1-\bar{x}) = 2\bar{x} - 1 \\ \implies & \hat{\theta}_{MoM} = \frac{2\bar{x}-1}{1-\bar{x}} \end{aligned}$$

   The method of moments estimator for θ is $\hat{\theta}_{MoM} = \frac{2\bar{X}-1}{1-\bar{X}}$

   Based on our data, $\bar{x} = 0.767$. Plugging this into our estimator,
   The method of moments estimate for θ is $\hat{\theta}_{MoM} = \frac{2(0.767)-1}{1-0.767} = 2.21$

2. Maximum Likelihood
   The Likelihood function is

$$L(\theta) = f(x_1, \cdots, x_n) = \prod_{i=1}^n f(x_i) = \prod_{i=1}^n (\theta+1)x_i^{\theta} = (\theta+1)^n \prod_{i=1}^n x_i^{\theta}$$

   The log-Likelihood functions is

$$\begin{aligned} l(\theta) = logL(\theta) = log\left((\theta+1)^n \prod_{i=1}^n x_i^{\theta}\right) &= nlog(\theta+1) + \theta log(x_1) + \theta log(x_2) + \dots \theta log(x_n) \\ &= nlog(\theta+1) + \theta\sum_{i=1}^n log(x_i) \end{aligned}$$

   We maximize the above log-Likelihood function by set it's first derivative with respect to θ equal to 0, and solving for θ.

$$\frac{d}{d\theta}l(\theta) = \frac{d}{d\theta}\left[nlog(\theta+1) + \theta\sum_{i=1}^{n} log(x_i)\right] = \frac{n}{\theta+1} + \sum_{i=1}^{n} log(x_i)$$

Setting $\frac{d}{d\theta}l(\theta)$ equal to 0, we get,

$$\begin{aligned}\frac{d}{d\theta}l(\theta) = \frac{n}{\theta+1} + \sum_{i=1}^{n} log(x_i) &\overset{set}{=} 0 \\ \implies \frac{n}{\theta+1} &= -\sum_{i=1}^{n} log(x_i) \\ \implies \theta + 1 &= \frac{-n}{\sum_{i=1}^{n} log(x_i)} \\ \implies \hat{\theta}_{MLE} &= \frac{-n}{\sum_{i=1}^{n} log(x_i)} - 1\end{aligned}$$

2^{nd} derivative test to check if we have maximum:

$$\frac{d^2}{d\theta^2}l(\theta)|_{\hat{\theta}} = \frac{d}{d\theta}\left[\frac{d}{d\theta}l(\theta)\right]|_{\hat{\theta}} = \frac{d}{d\theta}\left[\frac{n}{\theta+1} + \sum_{i=1}^{n} log(x_i)\right]|_{\hat{\theta}} = \frac{-n}{(\theta+1)^2}|_{\hat{\theta}} = \frac{-n}{(\hat{\theta}+1)^2} < 0$$

So, we have a maximum at $\hat{\theta}_{MLE}$.
Maximum likelihood estimator is $\hat{\theta}_{MLE} = \frac{n}{-\sum_{i=1}^{n} log(X_i)} - 1$

Plugging our data into the estimator (Note: That log represents the natural logarithm):
The maximum likelihood estimate for θ is $\hat{\theta}_{MLE} = \frac{-n}{\sum_{i=1}^{n} log(x_i)} - 1 = \frac{-3}{log(0.84)+log(0.51)+log(0.95)} - 1 = 2.34$