Lecture 13

Exponential Distribution

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Exponential Distribution

Exponential Distribution

Setup: The exponential distribution is commonly used to model waiting times between single occurrences, and lifetimes of electrical/mechanical devices.

Define the random variable

$$X = \text{“time between occurrences (rare events)”}$$

This random variable X follows an *exponential distribution*

$$X \sim Exp(\lambda)$$

where $\lambda > 0$ is the rate parameter.

Exponential PDF

- Probability Density Function (pdf)
  - $\text{Im(X)} = (0, \infty)$
  - $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

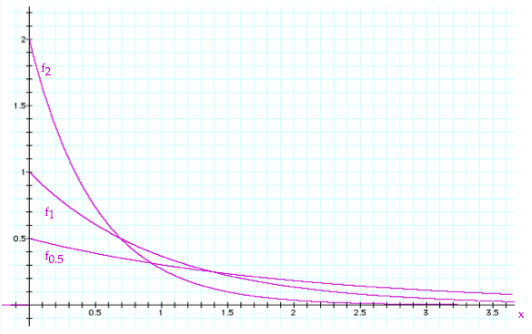

Figure 1: PDFs for exponential distributions with $\lambda = 0.5, 1, 2$

Exponential Distribution Cont.

- Cumulative distribution function (cdf)

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-\lambda t} & \text{for } t > 0 \end{cases}$$

- Expected Value: $E(X) = \frac{1}{\lambda}$

- Variance: $Var(X) = \frac{1}{\lambda^2}$

Examples

Exponential Distribution Example

Example 1: Suppose you create a website and are interested in modeling the time between hits to your website. On average, you receive 5 hits per minute.

- Hits to a website is a "rare" occurrence. Time between hits can be modeled using an exponential distribution

  Define the R.V: $X =$ time between hits to your website

  Distribution of X: $X \sim Exp(\lambda) \equiv Exp(?)$

- What value should we use for the rate parameter λ?

  $E(X) = \frac{1}{\lambda}$ is the average time between hits.
  We know there is an average of 5 hits/min. This means, on average, there is $\frac{1}{5}$ min/hit.
  $E(X) = \frac{1}{\lambda} = \frac{1}{5} \rightarrow \lambda = 5$

Exponential Distribution Example

$$X = \text{time between hits to your website}$$

$$X \sim Exp(5)$$

1. What is the expected time between hits?

2. What is the variance of time between hits?

3. What is the probability that we wait at most 40 seconds before someone visits your website?

Exponential Distribution Example

4. How long do we have to wait to observe a first hit with probability 0.9?

Memoryless Property

Memoryless Property of Exponential Distribution

- In the web page example, we said that we start to observe the web page at time point 0.
- Does the choice of this time point affect our analysis?
- If there is no hit in 1^{st} min, what is the probability that we get a hit in the next 40 seconds?

This is a conditional probability:

$$P(\text{obtain hit in 1 min and 40 sec} | \text{no hit in } 1^{st} \text{ min})$$
$$= P\left(X \leq \frac{5}{3} \middle| X > 1\right)$$

Memoryless Property of Exponential Distribution

$$
\begin{aligned}
&P(\text{obtain hit in 1 min and 40 sec} | \text{no hit in } 1^{st} \text{ min}) \\
&= P\left(X \leq \frac{5}{3} \middle| X > 1\right) \\
&= \frac{P\left(X \leq \frac{5}{3} \cap X > 1\right)}{P(X > 1)} \\
&= \frac{P(1 \leq X \leq \frac{5}{3})}{P(X > 1)} \\
&= \frac{F_X(\frac{5}{3}) - F_X(1)}{1 - F_X(1)} \\
&=
\end{aligned}
$$

This is exactly the same as $P(X \leq \frac{5}{3})$ which we calculated in Example 1 #3:

Memoryless Property of Exponential Dist. Cont.

$$P(Y \leq t + s | Y \geq s) = 1 - e^{-\lambda t} = P(Y \leq t)$$

- In other words, a random variable with an exponential distribution "forgets" about its past.
- This phenomena is called the *memoryless property*.
- An electrical/mechanical device whose lifetime we model as an exponential variable therefore *stays as good as new* until it suddenly breaks, i.e. we assume that there's no aging process.
- Exponential distribution is the only continuous distribution that has this property.
- Geometric distribution is a discrete distribution that also has this property.