Lecture 13

Exponential Distribution

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Exponential Distribution

Exponential Distribution

Setup: The exponential distribution is commonly used to model waiting times between single occurrences, and lifetimes of electrical/mechanical devices.

Define the random variable

$$X = \text{"time between occurrences (rare events)"}$$

This random variable X follows an *exponential distribution*

$$X \sim Exp(\lambda)$$

where $\lambda > 0$ is the rate parameter.

Exponential PDF

- Probability Density Function (pdf)
  - $\text{Im}(X) = (0, \infty)$
  - $f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

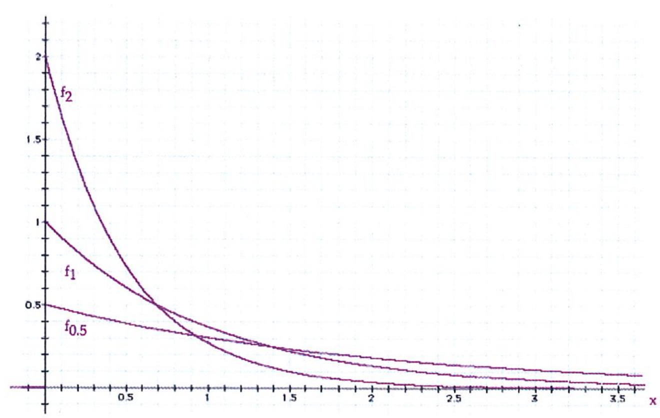

Figure 1: PDFs for exponential distributions with $\lambda = 0.5, 1, 2$

Exponential Distribution Cont.

- Cumulative distribution function (cdf)

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1 - e^{-\lambda t} & \text{for } t > 0 \end{cases}$$

- Expected Value: $E(X) = \frac{1}{\lambda}$

- Variance: $Var(X) = \frac{1}{\lambda^2}$

Examples

Exponential Distribution Example

Example 1: Suppose you create a website and are interested in modeling the time between hits to your website. On average, you receive 5 hits per minute.

- Hits to a website is a "rare" occurrence. Time between hits can be modeled using an exponential distribution

  Define the R.V: X = time between hits to your website

  Distribution of X: $X \sim Exp(\lambda) \equiv Exp(?)$ $\equiv Exp(5)$

- What value should we use for the rate parameter λ?

  $E(X) = \frac{1}{\lambda}$ is the average time between hits.

  We know there is an average of 5 hits/min. This means, on average, there is $\frac{1}{5}$ min/hit.

  $E(X) = \frac{1}{\lambda} = \frac{1}{5} \rightarrow \lambda = 5$

5 hits/min

$\Rightarrow$ 1 min/5 hits

$\Rightarrow$ 1/5 min/hit

Exponential Distribution Example

X = time between hits to your website

$X \sim Exp(5)$

$\boxed{\text{PDF}}$

$$f(x) = \begin{cases} \lambda e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

$$= \begin{cases} 5e^{-5x} & \text{for } x > 0 \\ 0 & \text{o.w.} \end{cases}$$

1. What is the expected time between hits?

   $E(X) = 1/\lambda = 1/5$

2. What is the variance of time between hits?

   $Var(X) = 1/\lambda^2 = 1/5^2 = 1/25$

3. What is the probability that we wait at most 40 seconds before someone visits your website?

   $= \frac{2}{3}$ min

   $P(X \leq 2/3) = F_X(2/3)$

   $= 1 - e^{-5(2/3)}$

   $= 0.9643$

$\boxed{\text{CDF}}$

$$F_X(t) = \begin{cases} 0 & \text{for } t \leq 0 \\ 1-e^{-\lambda t} & \text{for } t > 0 \end{cases}$$

$$= \begin{cases} 0 & \text{for } t \leq 0 \\ 1-e^{-5t} & \text{for } t > 0 \end{cases}$$

Exponential Distribution Example

4. How long do we have to wait to observe a first hit with probability 0.9?

Find the "t" that makes $P(X \leq t) = 0.9$

$$\Rightarrow \quad P(X \leq t) = 0.9$$

$$\Rightarrow \quad F_X(t) = 0.9$$

$$\Rightarrow \quad 1 - e^{-5t} = 0.9$$

$$\Rightarrow \quad e^{-5t} = 0.1$$

$$\Rightarrow \quad \ln(e^{-5t}) = \ln(0.1)$$

$$\Rightarrow \quad -5t = \ln(0.1)$$

$$\Rightarrow \quad t = \frac{-\ln(0.1)}{5} = 0.4605 \text{ min}$$

Memoryless Property

Memoryless Property of Exponential Distribution

- In the web page example, we said that we start to observe the web page at time point 0.
- Does the choice of this time point affect our analysis?
- If there is no hit in 1^{st} min, what is the probability that we get a hit in the next 40 seconds?

This is a conditional probability:

$$P(\text{obtain hit in 1 min and 40 sec}|\text{no hit in } 1^{st} \text{ min})$$
$$= P\left(X \le \frac{5}{3}\middle| X > 1\right)$$

Memoryless Property of Exponential Distribution

$$P(\text{obtain hit in 1 min and 40 sec}|\text{no hit in } 1^{st} \text{ min})$$
$$= P\left(X \le \frac{5}{3}\middle| X > 1\right)$$
$$= \frac{P(X \le \frac{5}{3} \cap X > 1)}{P(X > 1)}$$
$$= \frac{P(1 \le X \le \frac{5}{3})}{P(X > 1)}$$
$$= \frac{F_X(\frac{5}{3}) - F_X(1)}{1 - F_X(1)}$$
$$= 0.9643$$

This is exactly the same as $P(X \le \frac{\cancel{5}\,2}{3})$ which we calculated in Example 1 #3:

$$P(X \le 1 + \underline{2/3} \mid X > 1) = P(X \le 2/3)$$

Memoryless Property of Exponential Dist. Cont.

$$P(Y \leq t + s | Y \geq s) = 1 - e^{-\lambda t} = P(Y \leq t)$$

- In other words, a random variable with an exponential distribution "forgets" about its past.
- This phenomena is called the *memoryless property*.
- An electrical/mechanical device whose lifetime we model as an exponential variable therefore *stays as good as new* until it suddenly breaks, i.e. we assume that there's no aging process.
- Exponential distribution is the only continuous distribution that has this property.
- Geometric distribution is a discrete distribution that also has this property.