Exam 2 – Wed, Oct 23
Coverage: Lecture 5 – 10
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(front & back)
Bring calculator

Lecture 14

Gamma Distribution

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Gamma Distribution

Gamma Distribution

Setup: The gamma distribution is commonly used to model the total time for a procedure composed of α independent occurrences, where the time between each occurrence follows $Exp(\lambda)$

$$\underbrace{\boxed{\text{Exp} \mid \text{Exp} \mid \text{Exp} \mid \text{Exp} \mid \text{Exp}}}_{\text{Gamma}}$$

If a random variable follows a *Gamma distribution*,

$$X \sim Gamma(\alpha, \lambda)$$

λ lambda
α alpha

where $\lambda > 0$ is there rate parameter, and $\alpha > 0$ is the shape parameter

- Probability Density Function (pdf)
  - $Im(X) = (0, \infty)$
  - $f(x) = \begin{cases} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} & \text{for } x > 0 \\ 0 & \text{otherwise} \end{cases}$

wont use this directly →

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$ is called the "gamma function".

Gamma PDF

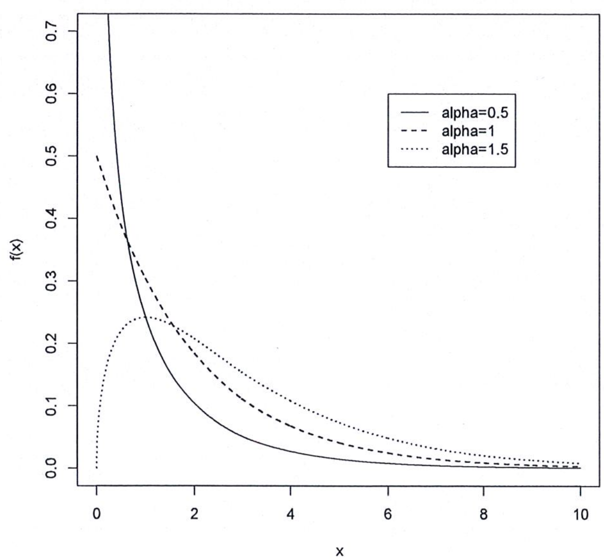

Figure 1: PDFs for gamma distribution with fixed λ and $\alpha = 0.5, 1, 1.5$

Gamma Distribution Summary

- Cumulative distribution function (cdf)

won't use this directly →

$$F_X(t) = \int_0^t f(x)dx = \frac{\lambda^\alpha}{\Gamma(\alpha)} \int_0^t x^{\alpha-1}e^{-\lambda x}dx$$

- Expected Value: $E(X) = \frac{\alpha}{\lambda}$
- Variance: $Var(X) = \frac{\alpha}{\lambda^2}$

Examples

Gamma Distribution Example

Example 1: Compilation of a computer program consists of 3 blocks that are processed sequentially, one after the other. Each block is independent of the other blocks, and takes Exponential time with mean of 5 minutes. We are interested in the total compilation time.

$$\underset{T_1}{\boxed{Exp(1/5)}}\underset{T_2}{\boxed{Exp(1/5)}}\underset{T_3}{\boxed{Exp(1/5)}}$$

- Total compilation time modeled using Gamma distribution.

  Define the R.V: T = total compilation time

  Distribution of T: $T \sim Gamma(\alpha, \lambda) \equiv Gamma(?, ?) \equiv Gamma(3, \frac{1}{5})$

- What value should we use for α and λ?
  - α is the number of independent occurrences (blocks) in the full procedure: $\alpha =$ 3
  - Time for each occurrence (call this "T_i") is exponential with mean 5 min. $E(T_i) = \frac{1}{\lambda} = 5 \rightarrow \lambda =$ 1/5

Gamma Distribution Example

R.V → T = total compilation time

$$T \sim Gamma(3, \frac{1}{5})$$

$\lambda = 1/5$

$\alpha = 3$

1. What is the expected value of total compilation time?

$$E(T) = \alpha/\lambda = 3/(1/5) = 15 \text{ min}$$

2. What is the variance of total compilation time?

$$Var(T) = \alpha/\lambda^2 = 3/(1/5)^2 = 75 \text{ min}$$

3. What is the probability for the entire program to be compiled in less than 12 minutes.

$$P(T < 12) = \int_{-\infty}^{12} f(x)dx = \int_0^{12} \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x} dx$$

$$= \int_0^{12} \frac{(1/5)^3}{\Gamma(3)} x^{3-1} e^{-x/5} dx$$

We won't do this →

$= \ldots$ (repeated integration by parts)

$=$ Gross

Poisson Approximation to Gamma Distribution

Gamma Distribution Example

- Could answer the previous question by using the Gamma CDF directly (requires repeated integration by parts)
- Instead, simplify Gamma probabilities by turning it into a Poisson problem!
- Turn a Gamma random variable into Poisson random variable using the Gamma-Poisson formula.

Gamma-Poisson Formula

For $T \sim \text{Gamma}(\alpha, \lambda)$ and $X \sim \text{Pois}(\lambda t)$,

1. $$P(T > t) = P(X < \alpha)$$

and

2. $$P(T \leq t) = P(X \geq \alpha)$$

Can turn a Gamma R.V into a Pois. R.V and use Poisson CDF Table/ poisson pmf to calculate probabilities

Gamma Distribution Example

(Back to Example 1 #3)

3. What is the probability for the entire program to be compiled in less than 12 minutes.

- Step 1: Define our Gamma random variable:
  $T \sim Gamma(\alpha, \lambda) \equiv Gamma(3, \frac{1}{5})$ $\alpha = 3$ $\lambda = 1/5$
  We want to know $P(T < t) = P(T < 12) = ?$ $t = 12$
- Step 2: Convert the Gamma R.V (T) into a Poisson R.V (X):
  $X \sim Pois(\lambda t) \equiv Pois(\frac{1}{5} \cdot 12) \equiv Pois(2.4)$ parameter for Poisson R.V
- Step 3: Use Gamma-Poisson formula: $P(T \le t) = P(X \ge \alpha)$

$$
\begin{aligned}
P(T < 12) &= P(T \le 12) && \text{since T is continuous R.V (Gamma)}\\
&= P(X \ge 3) && \text{Gamma/Poisson Formula}\\
&= 1 - P(X < 3)\\
&= 1 - F_X(2)\\
&= 1 - 0.5697 && \text{using Poisson CDF Table}\\
&= 0.4303
\end{aligned}
$$

Gamma Distribution Example

4. What is the probability that it takes at least 5 minutes to compile the entire program? $P(T \ge 5) = ?$ $t = 5$

STEP 1 : Define our Gamma R.V

$T \sim Gamma(3, 1/5)$ Want: $P(T \ge 5)$

α λ

STEP 2 : Convert the Gamma R.V (T) into Poisson R.V (X)

$X \sim Pois(\lambda t) \equiv Pois(1/5 \cdot 5) \equiv Pois(1)$

STEP 3 : Use Gamma-Poisson Formula: $P(T > t) = P(X < \alpha)$

$$
\begin{aligned}
P(T \ge 5) &= P(T > 5) && \text{since T is continuous R.V}\\
&= P(X < \alpha)\\
&= P(X < 3) && \text{since X is discrete R.V (Poisson)}\\
&= P(X \le 2)\\
&= F_X(2) && \text{using Appendix A Poisson CDF Table}\\
&= 0.9196
\end{aligned}
$$

(or $P(X \le 2) = P(X=0) + P(X=1) + P(X=2)$)