Lecture 16

Central Limit Theorem

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Central Limit Theorem (CLT)

Suppose $X_1, X_2, \ldots, X_n$ are iid random variables. For $i = 1, \ldots, n$,

$$X_i \overset{iid}{\sim} \text{distribution}$$

Any function of $\{X_i\}$ is also a random variable. Specifically,

- $S_n = \sum_{i=1}^{n} X_i$ is a R.V (with some distribution)
- $\overline{X_n} = \frac{\sum_{i=1}^{n} X_i}{n}$ is a R.V (with some distribution)

For large sample size n, the distribution of S_n and $\overline{X}$ both follow normal distributions!

Even without knowing the distribution of $\{X_i\}$, we can calculate probabilities for its sample mean and sample sum using the normal distribution. (extremely useful for real life problems)!

Idea of Central Limit Theorem

Central Limit Theorem (CLT)

- Sums and averages of RVs from *any* distribution have approximately normal distributions for large sample sizes

start w/ any dist ex: Exp, Unif, normal, gamma, etc that has a mean & variance

Central Limit Theorem (CLT)

Suppose $X_1, X_2, \ldots, X_n$ are iid random variables with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$ for $i = 1, \ldots, n$.
Define:

1. sample mean: $\overline{X_n} = \frac{\sum_{i=1}^n X_i}{n}$
2. sample sum: $S_n = \sum_{i=1}^n X_i$

Then, for *large n*,

$$\overline{X_n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

$$S_n \sim N(n\mu, n\sigma^2)$$

get the μ and σ^2 from original Xi distribution

How to Use CLT for Means

- For large n,

$$\overline{X_n} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

- How to calculate probabilities involving $\overline{X_n}$?
- Standardize $\overline{X_n}$ to turn it into a standard normal random variable Z, and use the $z-$table! (lecture notes ~~14~~) 15
- Standardize any normal random variable by subtracting its mean, and dividing by its standard deviation.

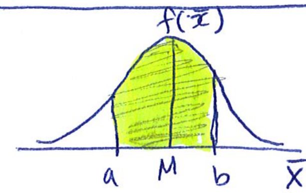
$$Z = \frac{\overline{X_n} - \mu}{\sigma/\sqrt{n}}$$

$$Z \sim N(0, 1)$$

std. normal dist use z-table to obtain CDF (probabilities)

How to Use CLT for Means Cont.

Before Standardizing

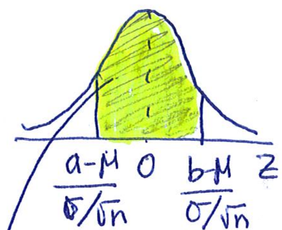


- Ex: $P(a < \overline{X_n} < b) = ?$
- Standardize all of the quantities involved in the above probability. Then use Z-table to obtain probabilities.

$$P(a < \overline{X_n} < b) = P\left(\frac{a-\mu}{\sigma/\sqrt{n}} < \frac{\overline{X_n}-\mu}{\sigma/\sqrt{n}} < \frac{b-\mu}{\sigma/\sqrt{n}}\right)$$
$$= P\left(\frac{a-\mu}{\sigma/\sqrt{n}} < Z < \frac{b-\mu}{\sigma/\sqrt{n}}\right)$$
$$= P\left(Z < \frac{b-\mu}{\sigma/\sqrt{n}}\right) - P\left(Z < \frac{a-\mu}{\sigma/\sqrt{n}}\right)$$
$$= \Phi\left(\frac{b-\mu}{\sigma/\sqrt{n}}\right) - \Phi\left(\frac{a-\mu}{\sigma/\sqrt{n}}\right)$$

use Z-table   use Z-table

After Standardizing



get this area using Z-table

How to Use CLT for Sums

- For large n,

$$S_n \sim N(n\mu, n\sigma^2)$$

- Standardize S_n by subtracting its mean, and dividing by its standard deviation.

$$Z = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n - n\mu}{\sigma\sqrt{n}}$$
$$Z \sim N(0,1)$$

- Then, use the $Z-$table to obtain desired probabilities.
- Ex:

$$P(S_n < a) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} < \frac{a - n\mu}{\sigma\sqrt{n}}\right)$$
$$= P\left(Z < \frac{a - n\mu}{\sigma\sqrt{n}}\right)$$
$$= \Phi\left(\frac{a - n\mu}{\sigma\sqrt{n}}\right)$$

Ex: $\Phi(1.66)$
Look up $z=1.66$ in margins of Z-table, and get $P(Z<1.66)$ inside Z-table

evaluates to just a number

Examples

Examples

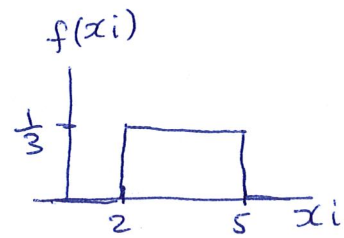
Example 1: The time you spend waiting for the bus each day has a uniform distribution between 2 minutes and 5 minutes. Suppose you wait for the bus every day for a month (30 days).

1. Let X_i = time spent waiting for the bus on the i^{th} day for $i = 1, \ldots, 30$.

   What is the distribution of each X_i?

   For $i = 1, \ldots, 30$ ($a = 2$, $b = 5$)

   $$X_i \overset{iid}{\sim} \text{Unif}(2, 5)$$



   What is it's expected value and variance?

   For each X_i (each day):

   $$E(X_i) = \frac{a+b}{2} = \frac{2+5}{2} = \frac{7}{2} = 3.5 = \text{"}\mu\text{"}$$

   $$Var(X_i) = \frac{(b-a)^2}{12} = \frac{(5-2)^2}{12} = \frac{9}{12} = 0.75 = \text{"}\sigma^2\text{"}$$

Examples

2. Let $\overline{X_n}$ be the average time spent waiting for the bus over the month. $\overline{X_n} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{\sum_{i=1}^{30} X_i}{30}$

   What is the (approximate) probability that the average time you spent waiting for the bus is less than 4 min? $P(\overline{X}_n < 4) = ?$

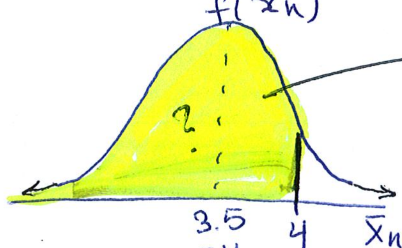
Now, we're interested in the R.V $\overline{X}_n = \sum_{i=1}^{30} X_i / 30$

Since n is large,

$$\overline{X}_n \sim N(\mu, \sigma^2/n) \equiv N(3.5, 0.75/30)$$

[μ & σ^2 always comes from dist. of original X_i's]

CLT for Means



$P(\overline{X} < 4) = ?$ can't get this directly

need to standardize and use z-table

Examples

standardize $\overline{X}$ into R.V Z

$$Z = \frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} = \frac{\overline{X}_n - 3.5}{\sqrt{0.75/30}} = \frac{\overline{X}_n - 3.5}{0.1581} \sim N(0,1)$$

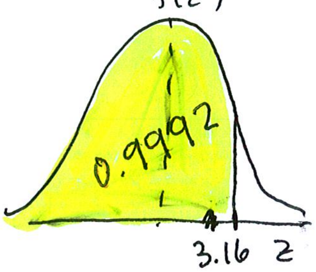
$$P(\overline{X}_n < 4) = P\left(\frac{\overline{X}_n - \mu}{\sigma/\sqrt{n}} < \frac{4-\mu}{\sigma/\sqrt{n}}\right)$$

$$= P\left(Z < \frac{4-3.5}{0.1581}\right)$$

$$= P(Z < 3.16)$$

$$= \Phi(3.16)$$

$$= 0.9992$$



use z-table

- Look up $z = 3.16$ in margins of z-table
- obtain $P(Z < 3.16)$ from inside z-table.

Examples

3. How much time do you expect to spend waiting for the bus in total for a month? Now we want $\sum_{i=1}^{30} X_i$ as our R.V

$$E\left(\sum_{i=1}^{30} X_i\right) = 30\,E(X_1) = 30 \cdot \mu = 30 \cdot 3.5 = 105$$

4. What is the (approximate) probability that you spend more than 2 hours waiting for a bus in total for a month?

2 hours = 120 min

Our new R.V is $S_n = \sum_{i=1}^{30} X_i$

Since n is large,

using CLT for sums →

$$S_n \sim N(n\mu, n\sigma^2)$$
$$\equiv N(30 \cdot 3.5,\ 30 \cdot 0.75)$$
$$\equiv N(105,\ 22.5)$$

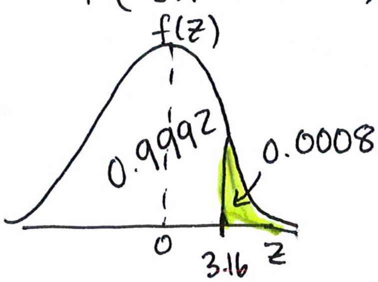
Examples

Want $P(S_n > 120) = ?$ can't get this directly need to standardize & use z-table

standardize

$$Z = \frac{S_n - n\mu}{\sqrt{n\sigma^2}} = \frac{S_n - n\mu}{\sigma\sqrt{n}} = \frac{S_n - 105}{\sqrt{22.5}}$$

$$P(S_n > 120) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} > \frac{120 - n\mu}{\sigma\sqrt{n}}\right)$$
$$= P\left(Z > \frac{120 - 105}{\sqrt{22.5}}\right)$$
$$= P(Z > 3.16)$$
$$= 1 - P(Z < 3.16)$$
$$= 1 - \Phi(3.16)$$
$$= 1 - 0.9992 = 0.0008$$



Examples

single X_i $\qquad E(X_i) = 1 = \mu$

$\qquad Var(X_i) = 0.5^2 = \sigma^2$

Example 2: Suppose an image has an expected size 1 megabyte with a standard deviation of 0.5 megabytes. A disk has 330 megabytes of free space. Is this disk likely to be sufficient for 300 independent images?

We're interested in the size of the sum of 300 images

$$S_n = \sum_{i=1}^{300} X_i$$

Since $n = 300$ is large, use CLT for sums,

$$S_n \sim N(n\mu, n\sigma^2)$$

$$= N(300 \cdot 1,\ 300 \cdot 0.5^2)$$

$$= N(300,\ 75)$$

We want to know if 330 MB is enough space

ie) $P(S_n \le 330) = ?$

Examples

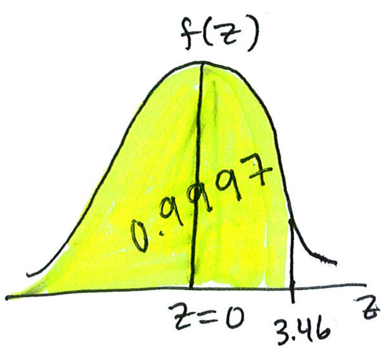
$$P(S_n \le 330) = P\left(\frac{S_n - n\mu}{\sigma\sqrt{n}} \le \frac{330 - n\mu}{\sigma\sqrt{n}} \right)$$

$$= P\left(Z \le \frac{330 - 300}{\sqrt{75}} \right)$$

$$= P(Z \le 3.46)$$

$$= \Phi(3.46)$$

$$= 0.9997$$



Examples

Example 3: An astronomer wants to measure the distance, d, from the observatory to a star. The astronomer plans to take n measurements of the distance and use the sample mean to estimate the true distance. From past records of these measurements the astronomer knows the standard deviation of a single measurement is 2 parsecs. How many measurements should the astronomer take so that the chance that his estimate differs by d by more than 0.5 parsecs is at most 0.05?

"true distance"

n — sample size

X_i = single measurement

For $i = 1, \ldots, n$

$Var(X_i) = 2^2 = \sigma^2$

For $i = 1, \ldots, n$

X_i = single measurement

$E(X_i) = \mu = d$

$Var(X_i) = \sigma^2 = 2^2 = 4$

Our new R.V is $\bar{X}_n = \sum_{i=1}^{n} X_i / n$

$$P(|\bar{X}_n - d| > 0.5) \leq 0.05$$

We want the minimum # of measurement (n) for this to be true

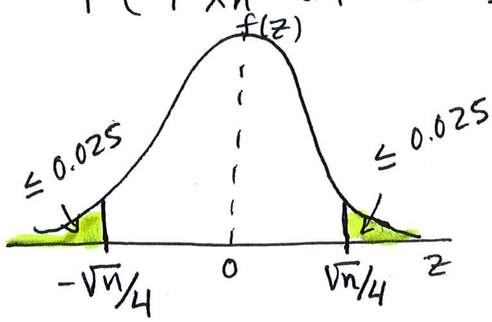
Examples

We know that $P(|\bar{X}_n - d| > 0.5) = P(\bar{X}_n - d > 0.5) + P(\bar{X}_n - d < -0.5)$

Using CLT for means, the distribution of $\bar{X}_n$ is

$$\bar{X}_n \sim N(\mu, \sigma^2/n) \equiv N(d, 4/n)$$

$$P(|\bar{X}_n - d| > 0.5) = P(\bar{X}_n - d > 0.5) + P(\bar{X}_n - d < -0.5)$$



$$= P\left(\frac{\bar{X}_n - d}{\sqrt{4/n}} > \frac{0.5}{\sqrt{4/n}}\right) + P\left(\frac{\bar{X}_n - d}{\sqrt{4/n}} < \frac{-0.5}{\sqrt{4/n}}\right)$$

$$= P(Z > 0.5/\sqrt{4/n}) + P(Z < -0.5/\sqrt{4/n})$$

$$= P(Z > \sqrt{n}/4) + P(Z < -\sqrt{n}/4)$$

$$= 2\,\Phi(-\sqrt{n}/4)$$

We need the smallest integer n such that

$$P(|\bar{X}_n - d| > 0.5) = 2\,\Phi(-\sqrt{n}/4) \leq 0.05$$

Examples

$$\Rightarrow \quad 2\,\Phi(-\sqrt{n}/4) \le 0.05$$

$$\Rightarrow \quad \Phi(-\sqrt{n}/4) \le 0.025$$

$$\Rightarrow \quad -\sqrt{n}/4 \le \Phi^{-1}(0.025)$$

$$\Rightarrow \quad -\sqrt{n}/4 \le -1.96$$

$$\Rightarrow \quad -\sqrt{n} \le 4\,(-1.96)$$

$$\Rightarrow \quad \sqrt{n} \ge 4\,(1.96)$$

$$\Rightarrow \quad n \ge (4 \cdot 1.96)^2 = 61.47$$

$\Phi^{-1}(0.025)$

- Find z in the margins s.t $P(Z \le z) = 0.025$
- Find 0.025 inside z-table, obtain corresponding z-score from margins

$\Rightarrow z = -1.96$

We need atleast $n = 62$ observations