Lecture 1

Basic Probability and Set Theory

Manju M. Johny

STAT 330 - Iowa State University

Course Info

Course Info

STAT 330: Probability and Statistics for Computer Science

General Info

- Instructor: Manju M. Johny
  - Email: mjohny@iastate.edu
  - Office Hours: 3:10-4pm Mon/Thurs, 3406 Snedecor Hall
- TA: ~~Quinglong~~ Qinglong Tian
  - Email: qltian@iastate.edu
  - Office Hours: 11-11:50am Tue/Fri, 3207 Snedecor Hall

Course Setup

1. Intro to probability & random variables (7-8 weeks)
2. Applications of probability ($\sim$ 3 weeks)
   - Markov chains
   - Queueing Systems
3. Statistics (4-5 weeks)

Introduction

Probability and Statistics

Definitions

- *Probability* is a mathematical theory for modeling processes where outcomes occur randomly
- *Statistics* is learning about the real world from data under the assumption that the data was generated by a random process

Goals

1. Probability: model and assign probabilities to outcomes
2. Statistics: use probability to draw conclusions

Random Experiment

Random Experiment

Definition

A *random experiment* is an experiment (or process) for which outcome cannot be predicted with complete certainty

Example 1: Various random experiments

Random Process

- A message can take two network routers to reach a recipient computer. We record the status of router 1, the status of router 2, and the status of the recipient computer, where the status is either up (U) or down (D).
- Record the time for a web page to respond.
- Roll a die and record the face up.
- Flip a coin until you get a head. Record all the faces that your obtain.

Components of random experiment

Outcome

lower case omega

Definition

The *outcome* (ω) is the result of an experiment

Example 2: Outcomes

- **Network Routers:** $\omega =$ (router 1 down, router 2 down, recipient computer up) = DDU
- **Access web page:** $\omega =$ 2.576 sec
- **Roll a die:** $\omega =$ 4
- **Toss coin until head:** $\omega =$ TTH

Sample space

capital Omega

Definition

The *sample space* (Ω) is the set of ALL possible outcomes

Example 3: Sample spaces

- **Network Routers:** $\Omega =$ DDD, DDU, DUD, UDD, UUD, UDU, DUU, UUU
- **Access web page:** $\Omega = (0, \infty)$
- **Roll a die:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
- **Toss coin until head:** $\Omega = \{H, TH, TTH, TTTH, \dots\}$

$|\Omega| = \#$ of outcomes in sample space.

For the network routers example, $|\Omega| = 8$

"cardinality" of Omega Ω

Types of sample space

Sample space can be ...

- finite → discrete
- countable infinite → discrete
- uncountable infinite → not discrete

Example 4: Discrete/not discrete sample spaces

- **Network Routers:**
  $\Omega = \{$DDD, DDU, DUD, UDD, UUD, UDU, DUU, UUU$\}$
  → discrete (finite)
- **Access web page:** $\Omega = (0, \infty)$
  → not discrete (uncountable infinite)
- **Roll a die:** $\Omega = \{1, 2, 3, 4, 5, 6\}$
  → discrete (finite)
- **Toss coin until head:** $\Omega = \{H, TH, TTH, TTTH, \cdots\}$
  → discrete (countable finite)

Event

Definition subset of sample space

An *event* (A, B, C, etc) is a collections of outcomes from the sample space that we are interested in. $A \subseteq \Omega$

Example 5: Events

- **Network Routers:** A message is transmitted successfully if at least one router is up and recipient computer is up.
  A = successful transmission = $\{$DUU, UDU, UUU$\}$
- **Access web page:** B = More than 10 seconds = $(10, \infty)$
- **Roll a die:** C = Roll even # = $\{2, 4, 6\}$
- **Toss coin until head:** D = getting 1st head in 2-5 tosses
  = $\{$TH, TTH, TTTH, TTTTH$\}$

Probability

Probability

How likely is an event?

- Let A be an event (set of outcomes from the sample space)
- Then, probability of event A is written as *P(A)*

Example 6:

- Consider event C as successful transmission in the Network Router example
- Suppose the chance that a message is successfully transmitted is 90%
- P(C) = 0.90

To calculate probability of events, start with understanding set theory

Set Theory

Set Notation

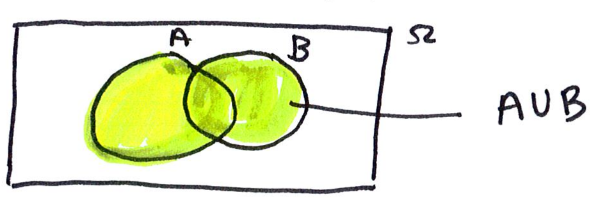
Review symbols $\in, \notin, \subset, \subseteq, \supset, \supseteq$.

- If x is an element of set B, this is denoted $x \in B$ x "in" B
- If y is not an element of set B, this is denoted $y \notin B$ y "not in" B
- If every element of set A is also an element of set B, then A is a subset of B. $A \subseteq B$

Let A and B be two events $\cdots$

- *Union* ($\cup$): $A \cup B$ is the event consisting of all outcomes in A or in B or in both.

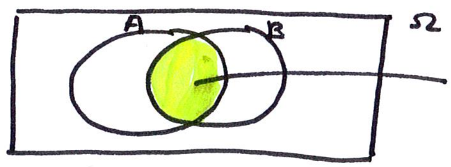
$$A \cup B = \{\omega \mid \omega \in A \text{ or } \omega \in B\}$$



Set Notation Cont.

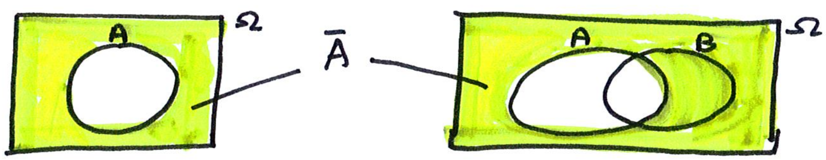
- *Intersection* ($\cap$): $A \cap B$ is the event consisting of all outcomes simultaneously in A and in B.

$$A \cap B = \{\omega \mid \omega \in A \text{ and } \omega \in B\}$$



- *Complement* ($\bar{A}$): The complement of an event A ($\bar{A}$) is the event consisting of all outcomes not in A. $\bar{A} = A^c$

$$\bar{A} = \{\omega \mid \omega \notin A\}$$



Set Notation Cont.

- *De Morgan's laws*: (added slide)

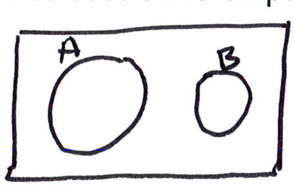
$$(\overline{A \cup B}) = \bar{A} \cap \bar{B} \qquad (\overline{A \cap B}) = \bar{A} \cup \bar{B}$$

- *Empty set* ($\emptyset$): $\emptyset$ is a set containing no elements, usually denoted by $\{\}$. The empty set is a subset of every set:

"null set"

$$\emptyset \subseteq A$$

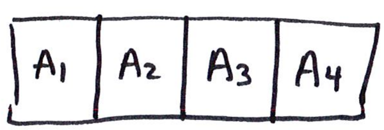
- *Disjoint/mutually exclusive sets*: Sets A, B are disjoint if their intersection is empty:



$$A \cap B = \emptyset$$

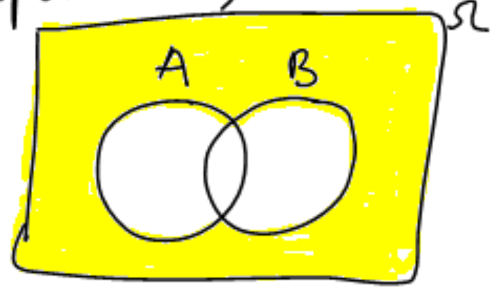
- no overlap
- no outcomes in common b/w A, B

- *Pairwise disjoint sets*: Sets $A_1, A_2, A_3, \cdots$ are pairwise disjoint if $A_i \cap A_j = \emptyset$ for any $i \neq j$

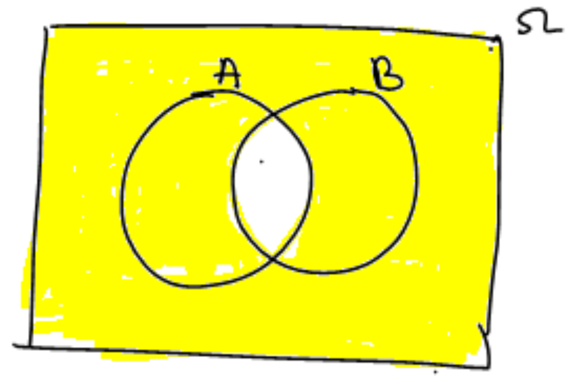


De Morgan's Law (Important)

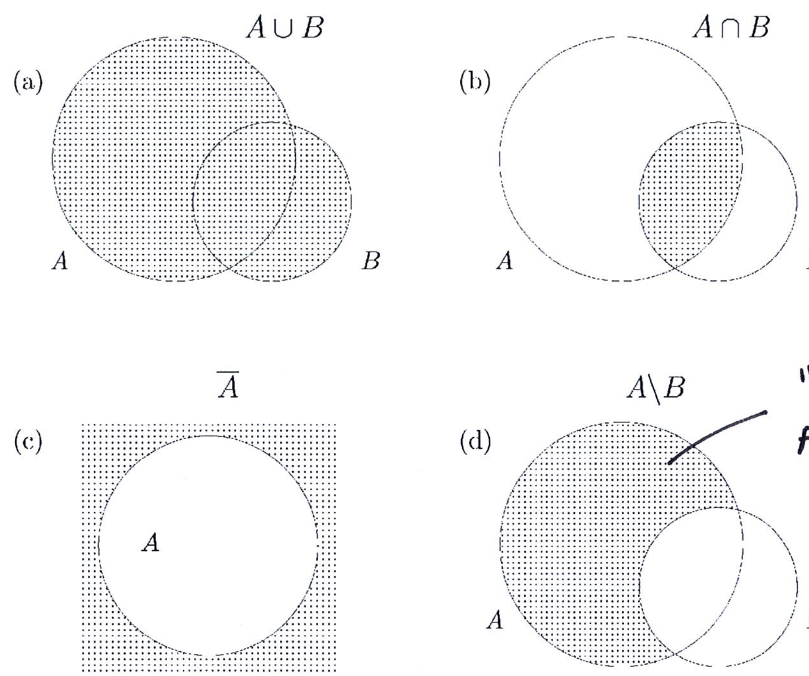
(1) $\overline{A \cup B} = \bar{A} \cap \bar{B}$



(2) $\overline{A \cap B} = \bar{A} \cup \bar{B}$



Venn Diagrams



"A not B"

$A \backslash B = A \cap \overline{B}$

Set Notation Cont.

Example 7:

$\Omega = \{1, 2, 3, 4, 5\}$

$A = \{1, 2, 3\}$

$B = \{2, 3, 4\}$

$C = \{4, 5\}$

1. $\bar{A} = \{4, 5\}$
2. $A \cup B = \{1, 2, 3, 4\}$
3. $A \cap B = \{2, 3\}$
4. $A \cap C = \emptyset$
5. Are A and B disjoint? no b/c $A \cap B \neq \emptyset$
6. Are A and C disjoint? yes b/c $A \cap C = \emptyset$
7. Are A,B,C pairwise disjoint? no
   at least one pair has elements in common

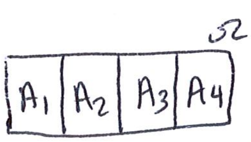
Kolmogorov's Axioms

Kolmogorov's Axioms

- Recall: $P(A)$ is the probability that event A occurs
- Want to assign probabilities to events as a measure of their likelihood of occurring
- A *probability model* is an assignment of numbers $P(A)$ to events $A \subseteq \Omega$ such that *Kolmogorov's axioms* are satisfied.

Kolmogorov's Axioms

1. $0 \leq P(A) \leq 1$ for all A (b/w 0 & 1 inclusive)
2. $P(\Omega) = 1$
3. If $A_1, A_2, A_3, \cdots$ are pairwise disjoint, then $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots = \sum_i P(A_i)$



Kolmogorov's Axioms Cont.

Kolmogorov's axioms...

- Give the logical framework that probability assignment must follow
- But don't tell us what probabilities to assign to events

Example 8: Draw a single card from a standard deck of playing cards: $\Omega = \{red, black\}$

Two different probability models are:

Model 1	Model 2
$P(\Omega) = 1$	$P(\Omega) = 1$
$P(red) = 0.5$	$P(red) = 0.3$
$P(black) = 0.5$	$P(black) = 0.7$

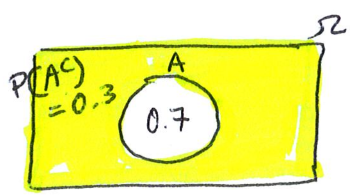
(b/c Kolmogorov's axioms are satisfied)

Both are *valid* probability models. However, real world experience tells us model 1 is more accurate for the scenario.

Consequences of Kolmogorov's Axioms

Let $A, B \subseteq \Omega$.

A. Probability of the Complementary Event:

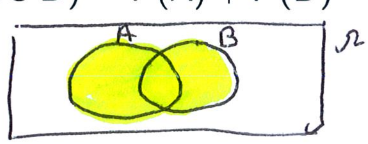


$$P(\overline{A}) = 1 - P(A)$$

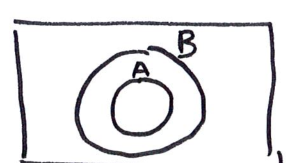
Corollary: $P(\emptyset) = 0$

B. Addition Rule of Probability

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



C. If $A \subseteq B$, then $P(A) \leq P(B)$.



Corollary: For any A, $P(A) \leq 1$.

Using Kolmogorov's axioms

There are 2 main approaches to assign probabilities to events at this point.

1. When we know events are disjoint (easy!).
    - Let A be a collection of of k outcomes $(\omega_1, \ldots, \omega_k)$ that are all *pairwise disjoint*.
    - Use Kolmogorov's axiom 3: $P(A) = P(\cup_{i=1}^{k} \omega_i) = \sum_{i=1}^{k} P(\omega_i)$.

Example 9: Roll a die. Suppose event A is rolling an even number. (Assume all numbers are equally likely $\rightarrow P(\omega) = \frac{1}{6}$ for all ω)
$\Omega = \{1, 2, 3, 4, 5, 6\}$
$A = \{2, 4, 6\}$

$$\begin{aligned} P(A) &= P(\text{"2" or "4" or "6"}) \\ &= P(\text{"2"} \cup \text{"4"} \cup \text{"6"}) \\ &= P(2) + P(4) + P(6) \\ &= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} \\ &= 3/6 = 1/2 \end{aligned}$$

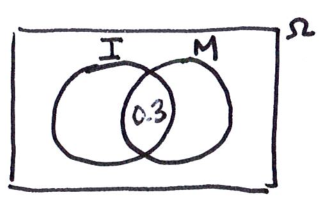
Kolmogorov's axiom 3 since sets are pairwise disjoint

Using Kolmogorov's axioms cont.

2. When events may or may not be disjoint (harder).
    - Start with known probability of some of the events.
    - Use this information and Kolmogorov's axioms to deduce probabilities of other events.
    - Drawing Venn diagrams will simplify the problem

Example 10: Suppose in a small college of 1000 students, 650 students own Iphones, 400 students own MacBooks, and 300 students own both.
Define events: I = "owns Iphone", and M = "owns MacBook".

Known
$P(I) = 0.65$
$P(M) = 0.40$
$P(I \cap M) = 0.30$

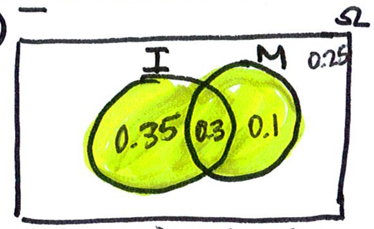


Using Kolmogorov's axioms cont.

$P(I) = 0.65 \quad P(M) = 0.4 \quad P(I \cap M) = 0.3$

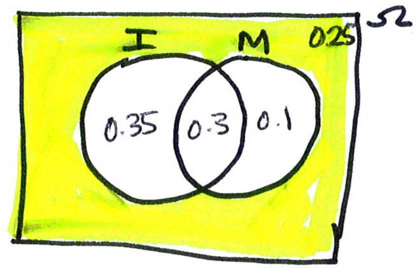
a. What is the probability of owning an Iphone or a MacBook?

$$P(I \cup M) = P(I) + P(M) - P(I \cap M)$$
$$= 0.65 + 0.4 - 0.3$$
$$= 0.75$$



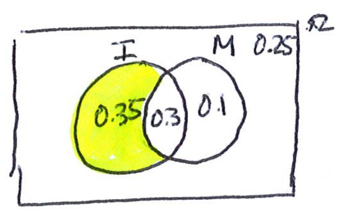
$P(I \cup M) = 0.75$

b. What is the probability of owning neither an Iphone nor a MacBook?



$$P(\bar{I} \cap \bar{M})$$
$$= P(\overline{I \cup M}) \quad \text{De Morgan's law}$$
$$= 1 - P(I \cup M) \quad \text{complement}$$
$$= 1 - 0.75$$
$$= 0.25$$

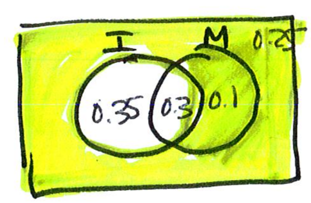
c. What is the probability of owning only an Iphone? (ie. owning an iphone and no MacBook)



$$P(I \setminus M) = P(I \cap \bar{M}) = ?$$
$$P(I) = P(I \cap \bar{M}) + P(I \cap M)$$
$$P(I \cap \bar{M}) = P(I) - P(I \cap M)$$
$$= 0.65 - 0.3$$
$$= 0.35$$

d. What is the probability of not owning an Iphone?

$$P(\bar{I}) = 1 - P(I)$$
$$= 1 - 0.65$$
$$= 0.35$$



What if we were given $P(A)$, $P(B)$, $P(\bar{A} \cap \bar{B})$

probability of neither A nor B

$$P(\bar{A} \cap \bar{B}) = P(\overline{A \cup B})$$

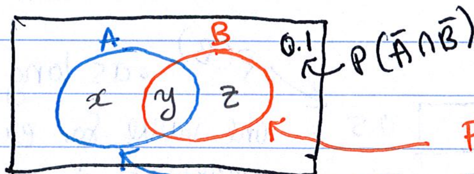
de morgan's law

Given

$P(A) = 0.7$

$P(B) = 0.5$

$P(\bar{A} \cap \bar{B}) = 0.1$



$P(B) = y + z = 0.5$

$P(A) = x + y = 0.7$

What is $P(A \cap B)$ $\quad P(A \cup B) = x + y + z = 0.9$

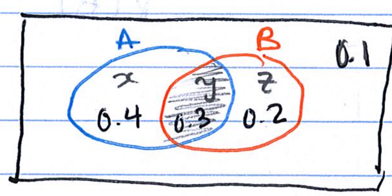
can solve using system of Eq's

$$x + y + z = 0.9$$
$$x + y = 0.7$$

$$0.7 + z = 0.9$$
$$z = 0.2$$

$$P(B) = y + z = 0.5$$
$$y + 0.2 = 0.5$$
$$y = 0.3$$

$$P(A) = x + y = 0.7$$
$$x + 0.3 = 0.7$$
$$x = 0.4$$



What is $P(A \cap B)$

$P(A \cap B) = 0.3$

<u>Table Method</u> (useful when dealing w/ 2 sets)

<u>Given</u>

$P(A) = 0.7$

$P(B) = 0.5$

$P(\bar{A} \cap \bar{B}) = 0.1$

	A	$\bar{A}$	
B			0.5
$\bar{B}$		0.1	
	0.7		1

$P(B)$

$P(A)$

- as long as you know one value for each ~~of~~ the margins & one cell value, you can deduce all other cell values
  - cells have to add to margin
  - margins (each side) adds to 1
  - all cells added together = 1

	A	$\bar{A}$	
B	0.3	0.2	0.5
$\bar{B}$	0.4	0.1	0.5
	0.7	0.3	1

$P(A \cap B) = 0.3$