Lecture 23

Method of Moments & Maximum Likelihood Estimation

Manju M. Johny

STAT 330 - Iowa State University

Estimating Parameters

2 General Methods for estimating parameters:

1. Method of moments estimation (MoM)
2. Maximum likelihood estimation (MLE)

Method of Moments (MoM)

Method of Moments (MoM)

also called population moment

Definition:

- The k^{th} *moment* of a R.V X is defined as $\mu_k = E(X^k)$
- The k^{th} *sample moment* is defined as $m_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k$

The method of moments (MoM) estimators for parameters are found by equating (known) sample moments to (unknown) population moments, and then solving for the parameters in terms of the data.

- If our model has more than one unknown parameter, we need to make equations with more than one moment.
- In general, need k equations to derive MoM estimators for k parameters.

MOM Cont.

To obtain MoM estimators for k parameters: Set the sample moments (m_k) equal to population moments (μ_k), and solve.

- $m_1 = \mu_1 \rightarrow \frac{1}{n}\sum x_i = E(X)$
- $m_2 = \mu_2 \rightarrow \frac{1}{n}\sum x_i^2 = E(X^2)$

  $\vdots$
- $m_k = \mu_k \rightarrow \frac{1}{n}\sum x_i^k = E(X^k)$

Note:

- MoM estimators may be biased
- Sometimes you can get estimates outside of parameter space

MoM Examples

MoM Example

Independent & identical

<u>Example 1</u>: Let $X_1, \dots, X_n \overset{iid}{\sim} Geo(p)$

Estimate one parameter $p \to$ need the first moment.

- 1^{st} (population) moment: $\mu_1 = E(X) = \frac{1}{p}$.
- 1^{st} sample moment is $m_1 = \frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}$

Set 1^{st} moment equal 1^{st} sample moment, and solve for p.

$$\frac{1}{p} = \bar{X} \to \underbrace{\hat{p}_{MoM} = \frac{1}{\bar{X}}}_{\text{MoM estimator for P}}$$

MoM Examples Cont.

<u>Example 2</u>: Let $X_1, \dots, X_n \overset{iid}{\sim} N(\overset{\text{unknown}}{\mu}, \overset{\text{unknown}}{\sigma^2})$

Estimate two parameters $\to$ need first two moments

Set the first two moments equal to the first two sample moments.

1. $\frac{1}{n}\sum_{i=1}^{n} X_i = E(X)$
2. $\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i^2}_{\text{sample moments}} = \underbrace{E(X^2)}_{\text{moment}}$

For our random variables, $E(X) = \mu$ and $Var(X) = \sigma^2$

From Eq 1, we have $\frac{1}{n}\sum X_i = E(X) = \mu$

$\to \hat{\mu}_{MoM} = \frac{1}{n}\sum X_i$

$\to \underbrace{\hat{\mu}_{MoM} = \bar{X}}_{\text{MoM estimator for } \mu}$

MoM Examples Cont.

$Var(X) = E(X^2) - E(X)^2 \to E(X^2) = Var(X) + E(X)^2 = \sigma^2 + \mu^2$

From Eq 2. we have:

$$\underbrace{\frac{1}{n}\sum_{i=1}^{n} X_i^2}_{\text{2nd sample moment}} = \underbrace{E(X^2) = \sigma^2 + \mu^2}_{\text{2nd moment}}$$

$$\to \sigma^2 = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \mu^2$$

$$\to \hat{\sigma}^2_{MoM} = \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \hat{\mu}^2_{MoM}$$

$$= \frac{1}{n}\sum_{i=1}^{n} X_i^2 - \bar{X}^2$$

$$= \underbrace{\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2}_{\text{is actually the biased version of sample variance.}}$$

Maximum Likelihood Estimation (MLE)

Likelihood Function

indep. & indentical

We have $X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$, where $f_X(x)$ has (unknown) parameter θ.

The model for our data is the *joint distribution* of $X_1, \ldots, X_n$

$$f_X(x_1, \ldots, x_n) = \prod_{i=1}^{n} f_X(x_i)$$

since X's are indep joint = product of marginals

same function

When the joint distribution is viewed as a function of the unknown parameter, it is referred to as the *likelihood function*

$$\mathcal{L}(\theta) = \prod_{i=1}^{n} f_X(x_i)$$

Likelihood Example

Example 3: $X_1 \ldots, X_n \overset{iid}{\sim} Pois(\lambda)$

The marginal distribution of each X_i is

$$f_X(x) = \frac{e^{-\lambda}\lambda^{x_i}}{x_i!}$$

pmf for a single X (marginal dist)

The joint distribution/likelihood function is

$$\begin{aligned}\mathcal{L}(\lambda) = f(x_1, \ldots, x_n) &= \prod_{i=1}^{n} f_X(x_i) \\ &= \prod_{i=1}^{n} \frac{e^{-\lambda}\lambda^{x_i}}{x_i!} \\ &= \frac{e^{-n\lambda}\lambda^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\end{aligned}$$

$\prod_{i=1}^{n} e^{-\lambda} = e^{-\sum_{i=1}^{n}\lambda} = e^{-n\lambda}$

$\prod_{i=1}^{n} \lambda^{x_i} = \lambda^{\sum_{i=1}^{n} x_i}$

Maximum Likelihood Estimation (MLE)

Definition

A *maximum likelihood estimator* $\hat{\theta}_{MLE}$ of θ is the function that "maximizes the likelihood (probability) of the data"

Thus, the MLE maximizes the joint distribution model or likelihood function:

$$\hat{\theta}_{MLE} = \underset{\theta \in \Theta}{\text{argmax}} \mathcal{L}(\theta) = \underset{\theta \in \Theta}{\text{argmax}} \prod_{i=1}^{n} f(x_i)$$

ie) find the θ that makes $L(\theta)$ the largest

→ find θ that is "most likely"

MLE Examples

Example 4: Flip a coin 10 times. Let X be the # of heads obtained. A reasonable model for X is $Bin(n = 10, p)$ where p is our unknown parameter that we would like to estimate.

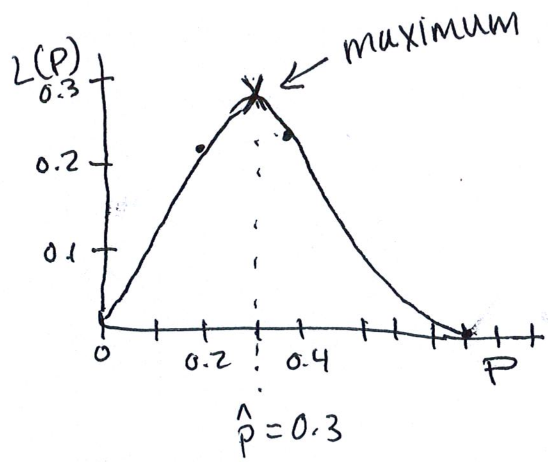
Suppose we observe the value $x = 3$. (only 1 data value).

Since there's only 1 data value, the likelihood/joint distribution is just the marginal distribution $f(x)$:

$$\begin{aligned}\mathcal{L}(p) = f(x) &= \binom{10}{x} p^x (1-p)^{10-x} \\ &= \binom{10}{3} p^3 (1-p)^{10-3} \\ &= \underbrace{120 p^3 (1-p)^7}_{\text{this is my likelihood function } L(p)}\end{aligned}$$

MLE Examples Cont.

What value of p maximizes the likelihood?



$p = 0.2 \rightarrow \mathcal{L}(0.2) = 0.201$

$p = 0.3 \rightarrow \mathcal{L}(0.3) = 0.267$

$p = 0.4 \rightarrow \mathcal{L}(0.4) = 0.215$

$p = 0.9 \rightarrow \mathcal{L}(0.9) = 0.000008$

Based on my data ($X = 3$)

MLE for p is $\hat{p}_{mle} = 0.3$

"the most likely value for my unknown parameter p"

General Calculation of MLE

- Maximizing the likelihood from $L(\theta)$ when there are multiple observed values becomes difficult. $L(\theta) = \prod_{i=1}^{n} f(x_i)$
- The common trick is to use the *log-likelihood function* instead:

$$\ell(\theta) = log\,\mathcal{L}(\theta)$$

where $\ell(\cdot)$ is the natural-log

$\rightarrow$ Since $\ell(\cdot)$ is increasing, the same θ that maximizes log-likelihood $\ell(\cdot)$ also maximizes the likelihood $\mathcal{L}(\theta)$

- Use calculus to find the maximum of $\ell(\theta)$

why

- using logs turns mult. problem into addition
- much easier to maximize

$L(\theta) = a \cdot b \cdot c$

$\log L(\theta) = \log(a \cdot b \cdot c)$

$= \log(a) + \log(b) + \log(c)$

General Calculation of MLE cont.

Finding MLE:

1. Find the likelihood function: $\mathcal{L}(\theta) = \prod_{i=1}^{n} f(x_i)$
2. Find the log-likelihood function: $\ell(\theta) = log\mathcal{L}(\theta)$
3. Take the first derivative: $\ell^{'}(\theta) = \frac{d}{d\theta}\ell(\theta)$
4. Set $\ell^{'}(\theta) = 0$ and solve for θ

   $\rightarrow$ this is your $\hat{\theta}_{MLE}$
5. Check if second derivative $\ell^{''}(\theta) < 0$ to make sure $\hat{\theta}_{MLE}$ is maximum

MLE Examples

MLE Examples

Example 5: Roll a (6-sided) die until you get a 6, and record the number of rolls. Repeat for 100 trials. For $i = 1, \dots 100$,

$$X_i = \# \text{ of rolls until you obtain a 6 in the } i^{th} trial$$

$$X_i \overset{iid}{\sim} Geo(p) \text{ and } f(x_i) = p(1-p)^{x_i - 1}$$

Data: Goal: estimate parameter p.

x	1	2	3	4	5	6	7	8	9
#	18	20	8	9	9	5	8	3	5

x	11	14	15	16	17	20	21	27	29
#	3	3	3	1	1	1	1	1	1

MLE Examples Cont.

1. Find the likelihood function $\mathcal{L}(p)$: (joint distribution)

$$\mathcal{L}(p) = \prod_{i=1}^{n} f(x_i) = \prod_{i=1}^{n} p(1-p)^{x_i - 1} = p^n (1-p)^{\sum_{i=1}^{n} x_i - n}$$

2. Find the log-likelihood function $\ell(p) = log\mathcal{L}(p)$:

$$\ell(p) = log\mathcal{L}(p)$$

recall
$\log(a^n) = n\log(a)$
$\log(a \cdot b) = \log a + \log b$

$$\ell(p) = \log L(p) = \log\left(p^n (1-p)^{\Sigma x_i - n}\right)$$
$$= \log(p^n) + \log\left((1-p)^{\Sigma x_i - n}\right)$$
$$= \underbrace{n \log(p) + (\Sigma x_i - n)\log(1-p)}_{\text{log-likelihood function}}$$

MLE Examples Cont.

3. Take the 1^{st} derivative w.r.t p: $\ell'(p)$:

$$\ell'(p) = \frac{d}{dp}\ell(p) = \frac{d}{dp} n log(p) + (\sum_{i=1}^{n} x_i - n) log(1-p)$$

$$= \frac{n}{p} + \frac{(\sum x_i - n)}{1-p}(-1)$$

$$= \frac{n}{p} - \frac{\sum x_i - n}{1-p}$$

MLE Example

4. Set $\ell'(p) = 0$ and solve for p:

$$\frac{d}{dp}\ell(p) \overset{set}{=} 0$$

$$\Rightarrow \frac{n}{p} - \frac{\sum x_i - n}{1-p} \overset{set}{=} 0$$

$$\Rightarrow \frac{n}{p} = \frac{\sum x_i - n}{1-p}$$

$$\Rightarrow \frac{1-p}{p} = \frac{\sum x_i - n}{n}$$

$$\Rightarrow \frac{1}{p} - 1 = \bar{x} - 1$$

$$\Rightarrow \frac{1}{p} = \bar{x}$$

$$\Rightarrow p = 1/\bar{x} \quad \Rightarrow \hat{p}_{mle} = \frac{1}{\bar{x}}$$

MLE Examples Cont.

5. 2^{nd} derivative test to confirm we have maximum:
$\frac{d^2}{dp^2}\ell(p)$

$$\frac{d^2}{dp^2}\ell(p) = \frac{d}{dp}\left[\frac{d}{dp}\ell(p)\right]\Big|_{p=\hat{p}}$$

$$= \frac{d}{dp}\left[\frac{n}{p} - \frac{\sum X_i - n}{1-p}\right]\Big|_{p=\hat{p}_{mle}}$$

$$= \frac{-n}{p^2} - - \frac{(\sum X_i - n)(-1)}{(1-p)^2}\Big|_{p=\hat{p}_{mle}}$$

$$= \frac{-n}{p^2} - \frac{\sum X_i - n}{(1-p)^2}\Big|_{\hat{p}_{mle}}$$

$$= \underbrace{\frac{-n}{\hat{p}^2}}_{\ominus} - \frac{\overbrace{\sum X_i - n}^{\geq 0}}{\underbrace{(1-\hat{p})^2}_{+}} < 0$$

$X \sim Geo(p)$
$Im(X) = \{1, 2, 3, \ldots\}$
So $\sum X_i$ is atleast $n \cdot 1 = n$
So $\sum X_i - n \geq 0$

MLE Example Cont.

So we have a maximum at $\hat{p}_{mle}$

Hence $\hat{p}_{mle} = \frac{1}{\bar{X}}$

Plug in the data into our MLE:

Data
$\sum X_i = 568$
$n = 100$
$\bar{X} = 5.68$

$$\hat{p}_{mle} = \frac{1}{\bar{X}} = \frac{1}{5.68} = 0.176$$