Lecture 24

Confidence Intervals

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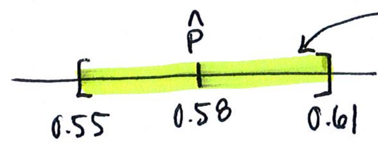
Confidence Intervals

Confidence Intervals

- MLE gives us a "point estimate" of the unknown parameter.
- But $\hat{\theta}$ probably won't *exactly* equal θ due to sampling error.
  $\rightarrow P(\theta = \hat{\theta}) = 0$
- Create a confidence interval to give range of reasonable values for the unknown parameter θ.

Example 1: Polling
Today's poll shows 58% of people favor the new bill. The margin of error is $\pm 3\%$.

The confidence interval for the proportion of people that favor the bill is $[0.55, 0.61]$.



where we think unknown parameter P is.

Confidence Interval

Definition

lower end point   upper end point

A random interval $[a, b]$ is a $(1-\alpha)100\%$ *confidence interval* for the parameter θ if it contains θ with probability $(1-\alpha)$

$$P(a \le \theta \le b) = 1 - \alpha$$

- $(1-\alpha)$ is called the confidence level
- When you estimate an unknown parameter θ, it should be accompanied by a confidence interval

90%, 95%, 99%

- Interpretation: We are $[(1-\alpha)\%]$ confident that the [insert population parameter + context] is between [insert interval + units].

Constructing Confidence Intervals

In this class, we will construct normal distribution based intervals.

Suppose we have an estimator $\hat{\theta}$ for unknown parameter θ.

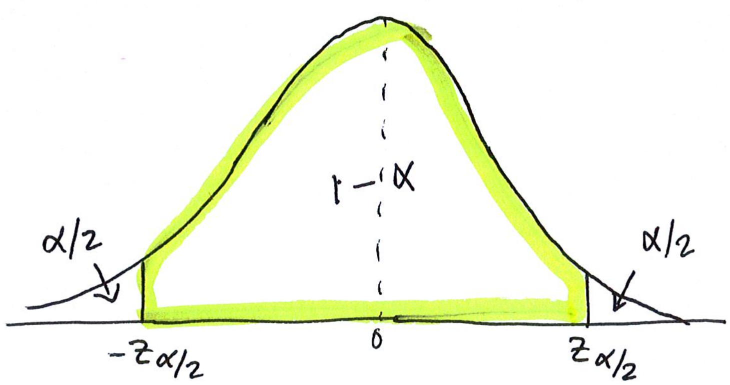
1. $\hat{\theta}$ is unbiased: $E(\hat{\theta}) = \theta$
2. $\hat{\theta}$ follows a normal distribution.

We can standardize $\hat{\theta}$ to get

$$Z = \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \sim N(0, 1)$$

where $\underbrace{SE(\hat{\theta})}_{\text{standard error}} = \sqrt{Var(\hat{\theta})}$ = standard deviation of $\hat{\theta}$

Constructing Confidence Intervals



Let $z_{\alpha/2}$ be the $1 - \frac{\alpha}{2}$ quantile of the standard normal distribution.

$$P\Big(-z_{\alpha/2} \le \frac{\hat{\theta} - \theta}{SE(\hat{\theta})} \le z_{\alpha/2}\Big) = 1 - \alpha$$

Constructing Confidence Intervals

Isolating θ in the middle, we get

$$P\left(\hat{\theta} - z_{\alpha/2}SE(\hat{\theta}) \leq \theta \leq \hat{\theta} + z_{\alpha/2}SE(\hat{\theta})\right) = 1 - \alpha$$

Thus, a $(1-\alpha)100\%$ confidence interval for θ is

$$\hat{\theta} \pm z_{\alpha/2}SE(\hat{\theta})$$

Common choices for α are $0.01, 0.05,$ and 0.1

$(1-\alpha)100\%$ 99% 95% 90%

confidence level

$(1-\alpha)100\%$	80	90	95	98	99
$z_{\alpha/2}$	1.282	1.645	1.96	2.326	2.576

Formula sheet

Constructing Confidence Intervals Cont.

We will make confidence intervals for four cases:

1. μ (population mean)
2. p (population proportion)
3. $\mu_1 - \mu_2$ (difference in population means)
4. $p_1 - p_2$ (difference in population proportions)

Confidence intervals for all 4 of the above cases can be constructed using normal distribution based inference.

Follow the same general procedure to construct these intervals.

$$\hat{\theta} \pm z_{\alpha/2} SE(\hat{\theta})$$

Confidence Interval for Mean

Confidence Interval for μ

Confidence interval for the population mean

$X_1, \ldots, X_n \overset{iid}{\sim} f_X(x)$ with $E(X_i) = \mu$ and $Var(X_i) = \sigma^2$

First, we estimate μ using the *statistic* $\bar{X}$. From CLT, we know

- $\bar{X} \sim N(\mu, \frac{\sigma^2}{n})$
- $SE(\bar{X}) = \sqrt{Var(\bar{X})} = \sqrt{\frac{\sigma^2}{n}} = \frac{\sigma}{\sqrt{n}}$

A $(1-\alpha)100\%$ confidence interval for μ is

$$\bar{X} \pm z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$$

In most cases, the population standard deviation σ will be unknown. Replace σ with the sample standard deviation s.

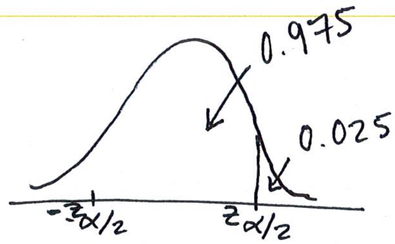
$$\bar{X} \pm z_{\alpha/2}\frac{s}{\sqrt{n}}$$

Formula sheet

Confidence Interval for μ Cont.

If we want a 95% confidence interval, then

$$1 - \alpha = 0.95$$
$$\rightarrow \alpha = 0.05$$
$$\rightarrow \alpha/2 = 0.025$$



$z_{\alpha/2} = z_{0.025}$ is the 0.975^{th} quantile of the $N(0,1)$ distribution.
$\rightarrow$ Using the $z - table$, we get $z_{0.025} = 1.96$

or use the table on slide 6 for common $z_{\alpha/2}$ values.

The 95% confidence interval for μ is

$$\bar{X} \pm 1.96 \frac{\sigma}{\sqrt{n}} \qquad \text{when } \sigma \text{ is known}$$
$$\bar{X} \pm 1.96 \frac{s}{\sqrt{n}} \qquad \text{when } \sigma \text{ is unknown}$$

Example

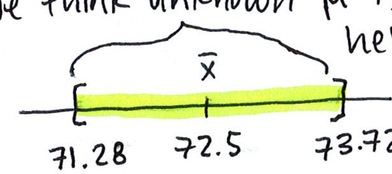
Example 2: A random sample of 50 batteries were taken for a particular brand. For the sample, the mean lifetime is 72.5 hours and variance is 19.3 hours^2. Find a 95% confidence interval for the true mean lifetime of batteries from that particular brand.

$n = 50$
$\bar{x} = 72.5$
$s^2 = 19.3$
$\Rightarrow s = \sqrt{19.3}$
$= 4.39$

$$\bar{X} \pm z_{\alpha/2}\, SE(\bar{X})$$
$$\Rightarrow \bar{X} \pm z_{\alpha/2} \cdot \frac{s}{\sqrt{n}}$$
$$\Rightarrow 72.5 \pm 1.96 \cdot \left(\frac{4.39}{\sqrt{50}}\right)$$
$$\Rightarrow 72.5 \pm \underbrace{1.2168}_{\text{margin of error}}$$
$$\Rightarrow [71.28, 73.72]$$

95% confidence
$z_{\alpha/2} = 1.96$

we think unknown μ is here



Example Cont.

Interpret the CI

confidence level

parameter + context

We are 95% confident that the true mean life time of the battery is between 71.28 and 73.72 hrs.

confidence interval (+ units)

Selecting Sample Size for Means

How to decide sample size? (Not on Final Exam)

- Can choose the sample size n to obtain a desired level of confidence & width for our confidence interval.
- Margin or error (Δ) is half the width of the confidence interval
  margin of error $= \Delta = z_{\alpha/2} SE(\bar{X}) = z_{\alpha/2}\frac{\sigma}{\sqrt{n}}$
- The bigger the sample size, the smaller the standard error of the estimator, and smaller the size of our interval

To attain a particular margin of error Δ, we need a sample size

$$n \geq \left(\frac{z_{\alpha/2}\sigma}{\Delta}\right)^2$$

Confidence Interval for Proportion

Confidence Interval for p

Confidence interval for the population proportion

- In this scenario, we want to estimate the proportion of population belonging to a particular category.
- Any individual in the population either belongs to the category of interest ("1"), or they don't ("0").
- Thus, we can think of each random variable X as a Bernoulli distribution with unknown parameter p
- We ultimately want to estimate and find a confidence interval for p.

"success" is being in category of interest

$p = P(\text{"success"}) = P(\text{being in category of interest})$

Confidence Interval for p Cont.

$X_1, \ldots, X_n \overset{iid}{\sim} Bern(p)$

First, estimate p using the *statistic* $\hat{p} = \frac{\sum X_i}{n}$ = sample proportion.

- $E(\hat{p}) = E(\frac{\sum_{i=1}^n X_i}{n}) = \frac{1}{n}E(\sum_{i=1}^n X_i) = \frac{1}{n}np = p$ (unbiased)
- $Var(\hat{p}) = Var(\frac{\sum X_i}{n}) = \frac{1}{n^2}Var(\sum X_i) = \frac{np(1-p)}{n^2} = \frac{p(1-p)}{n}$

  $\rightarrow SE(\hat{p}) = \sqrt{Var(\hat{p})} = \sqrt{\frac{p(1-p)}{n}}$

Since $\hat{p}$ is the mean of the Bernoulli $X's$, CLT for means applies

$$\hat{p} \sim N(p, \frac{p(1-p)}{n})$$

Thus a $(1-\alpha)100\%$ confidence interval for p is

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

formula sheet

Example

Example 3: In a random sample of 1000 U.S. adults, 38.8% stated they believed in the existence of ghosts. Find a 90% confidence interval for the population proportion of all U.S. adults who believe in the existence of ghosts.

$n = 1000$

$\hat{p} = 0.388$

90% confidence

$\Rightarrow z_{\alpha/2} = 1.645$

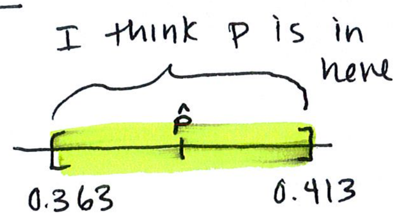
(table slide 6)

$$\hat{p} \pm z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$$

$$\Rightarrow 0.388 \pm 1.645\sqrt{\frac{0.388(1-0.388)}{1000}}$$

$$\Rightarrow 0.388 \pm 0.0253$$

$$\Rightarrow [0.363, 0.413]$$



We are 90% confident that the population proportion of all US adults that believe in ghosts is between 0.363 and 0.413.

Selecting Sample Size for Proportions

How to decide sample size? (Not on Final Exam)

- Just as before, we can select the sample size based on how large we want our margin or error to be

  margin or error $= \Delta = z_{\alpha/2} SE(\hat{p}) = z_{\alpha/2}\sqrt{\frac{\hat{p}(1-\hat{p})}{n}}$
- Issue: We haven't taken the sample, so we don't know $\hat{p}$
- Solution: Use $\hat{p} = 0.5$ for most conservative sample size

  $\Delta = z_{\alpha/2}\sqrt{\frac{0.5 \cdot 0.5}{n}} = z_{\alpha/2}\sqrt{\frac{0.5^2}{n}}$

To attain a particular margin of error Δ, we need a sample size

$$n \geq \left(\frac{z_{\alpha/2} \cdot 0.5}{\Delta}\right)^2$$

Sample Size Calculation Cont.

(Not on Final Exam)

Example 4: Political polls typically use 95% confidence and report margin of errors of 3%: $\hat{p} \pm 0.03$.
What sample size do we need to for such a poll?

Δ = margin of error $= 0.03$

95% confidence $\Rightarrow z_{\alpha/2} = 1.96$

$$n \geq \left(\frac{z_{\alpha/2} \cdot 0.5}{\Delta}\right)^2 = \left(\frac{1.96 \cdot 0.5}{0.03}\right)^2 = 1067.11$$

(always round up)

We want atleast 1068 people in our study.