Lecture 3

Conditional Probability & Independence

Manju M. Johny

STAT 330 - Iowa State University

Contingency Table

Contingency Table

Definition (Two Way Table)

A *contingency table* gives the distribution of 2 variables.

"break down"

Example 1: Suppose in a small college of 1000 students, 650 students own Iphones, 400 students own MacBooks, and 300 students own both.

Define events: $I =$ "owns Iphone", and $M =$ "owns MacBook".

$|I \cap M|$: # of students that own Iphone and Mac

Phone \ Computer	M	$\overline{M}$	Total
I	300	?	650
$\overline{I}$	?	?	?
Total	400	?	1000

$|I|$ # who own Iphone

Total # in sample space Ω $= |\Omega|$

$|M|$ # who own Mac

Contingency Table

- Fill in rest of table
- Inner cells add to margins
- margin sides add to total.

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

Marginal Probability

Marginal Probability

Definition

The *marginal probability* is the probability of a variable. It can be obtained from the *margins* of contingency table.

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

What is the probability of owning a Mac? (ie marginal probability of owning a Mac)

$P(M) = \frac{400}{1000} = 0.40$

$P(M) = \frac{|M|}{|\Omega|} = \frac{400}{1000} = 0.40$

Conditional Probability

Conditional Probability

Does knowing someone owns an Iphone change the probability they own a Mac?

Informally, conditional probability is updating the probability of an event given information about another event.

If we *know* that someone owns an Iphone, then we can narrow our sample space to just the "owns Iphone" case (highlighted blue row) and ignore the rest!

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

Conditional Probability Cont.

What is the probability of owning a Mac *given* they own an Iphone?

Phone \ Computer	M	$\overline{M}$	Total
I	300	350	650
$\overline{I}$	100	250	350
Total	400	600	1000

$P(M|I) = \frac{300}{650} = 0.46$

Conditional Probability Cont.

Definition

The *conditional probability* of event A given event B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

provided $P(B) \neq 0$.

It can be obtained from the *rows/columns* of contingency table.

Back to Example 1 . . .

What is the probability of owning a Mac *given* they own an Iphone?

$$P(M|I) = \frac{P(I \cap M)}{P(I)} = \frac{0.3}{0.65} = 0.46$$

Consequences of Conditional Probability

The definition of conditional probability gives useful results:

1.

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \rightarrow P(A \cap B) = P(B)P(A|B)$$

2.

$$P(B|A) = \frac{P(A \cap B)}{P(A)} \rightarrow P(A \cap B) = P(A)P(B|A)$$

This gives us two additional ways to calculate probability of intersections. Putting it together ...

$$\boxed{P(A \cap B) = P(A|B)P(B) = P(B|A)P(A)}$$

Probability Calculations

Probability Calculations

A contingency table can also be written with probabilities instead of counts. This is called a *probability table*.

Inner cells give "joint probabilities" → probability of intersections

- $P(A \cap B), P(\overline{A} \cap B)$, etc

Margins give "marginal probabilities" → probability of variables

- $P(A), P(B), P(\overline{A})$, etc

$P(I \cap M) = \frac{|I \cap M|}{|\Omega|} = \frac{300}{1000} = 0.3$

Phone \ Computer	M	$\overline{M}$	Total
I	0.30	0.35	0.65
$\overline{I}$	0.10	0.25	0.35
Total	0.40	0.60	1

$P(I) = \frac{|I|}{|\Omega|} = 0.65$

$P(\Omega) = \frac{|\Omega|}{|\Omega|} = \frac{1000}{1000} = 1$

Probability Calculations Cont.

Phone \ Computer	M	$\overline{M}$	Total
I	0.30	0.35	0.65
$\overline{I}$	0.10	0.25	0.35
Total	0.40	0.60	1

$P(\overline{I}) =$ 0.35

$P(M) =$ 0.40

$P(\overline{I} \cap M) =$ 0.10

$P(M|\overline{I}) = \frac{P(M \cap \overline{I})}{P(\overline{I})} = \frac{0.10}{0.35} = 0.29$

probability of owning mac given they don't own iphone

$P(\overline{I}|M) = \frac{P(\overline{I} \cap M)}{P(M)} = \frac{0.10}{0.40} = 0.25$

probability of not owning iphone given they own mac.

Independence

Independence of Events

In Example 1, knowing an event occurred changed the probability of another event occurring.

However, sometimes knowing an event occurs *doesn't change* the probability of the other event.
In this case, we say the events are *independent*.

Definition

Events A and B are *independent* if ...

1. $P(A \cap B) = P(A)P(B)$

   or equivalently

2. $\underbrace{P(A|B) = P(A)}$ if $P(B) \neq 0$

knowing B occurs has
no impact on probability
of A

Independence of Events Cont.

Example 2: Check if events are independent

Is owning an Iphone and owning MacBook independent?
Recall that $P(I) = 0.65$, $P(M) = 0.4$, $P(I \cap M) = 0.35$

→ check whether $P(I \cap M) \stackrel{?}{=} P(I)P(M)$

- $P(I \cap M) = 0.35$
- $P(I)P(M) = (0.65)(0.40) = 0.26$

Since $P(I \cap M) \neq P(I)P(M)$,

I, M are not independent

Independence of Events Cont.

Example 3: Using independence to simplify calculations
If A, B independent $\rightarrow P(A \cap B) = P(B)P(A|B) = P(B)P(A)$

Roll a die 4 times. Assuming that rolls are independent, what is the probability of obtaining at least one '6'?

$P(\text{at least 1 '6'}) = 1 - P(\text{No '6's})$
$= 1 - P(\text{no '6' on roll } 1 \cap \text{no '6' on roll } 2 \cap \cdots \cap \text{no '6' on roll } 4)$

Indep.

$= 1 - \left[\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)\right]$

$= 1 - \left[(5/6)^4\right]$

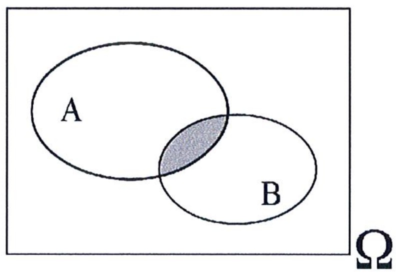
$= 0.518$

Independent vs. Disjoint

Independent ≠ Disjoint!!!

Completely different concepts!

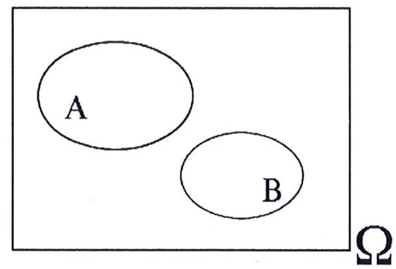
Independent:



$$P(A \cap B) = P(A)P(B)$$

overlap area = P(A)P(B)

Disjoint:



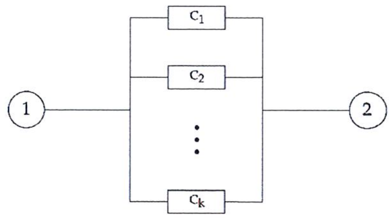
$$P(A \cap B) = P(\emptyset) = 0$$

no overlap

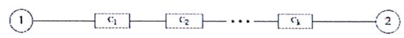
System Reliability

Application: System Reliability

Parallel: A parallel system consists of k components $(c_1, \ldots, c_k)$ arranged such that the system works if and only if at least one of the k components functions properly.

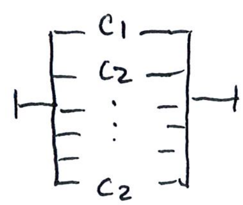


Series: A series system consists of k components $(c_1, \ldots, c_k)$ arranged such that the system works if and only if ALL components function properly.



Reliability: Reliability of a system is the probability that the system works.

Reliability of Parallel System



Example 4:

Let $c_1, \ldots, c_k$ denote the k components in a *parallel* system. Assume the k components operate independently, and

- $P(c_j \text{ works }) = p_j$. What is the reliability of the system?
- $P(c_j \text{ fails }) = P(\overline{c_j \text{ works}}) = 1 - P(c_j \text{ works}) = 1 - p_j$

$$
\begin{aligned}
P(\text{system works}) &= P(\text{at least one component works}) \\
&= 1 - P(\text{all components fail}) \\
&= 1 - P(c_1 \text{ fails} \cap c_2 \text{ fails} \cap \cdots \cap c_k \text{ fails}) \\
&= 1 - [P(c_1 \text{ fails}) P(c_2 \text{ fails}) \cdots P(c_k \text{ fails})] \\
&= 1 - \prod_{j=1}^{k} P(c_j \text{ fails}) \\
&= 1 - \prod_{j=1}^{k} (1 - p_j)
\end{aligned}
$$

Indep

Reliability of Series System

$\vdash c_1 - c_2 - \cdots - c_k \dashv$

Example 5:

Let $c_1, \ldots, c_k$ denote the k components in a *series* system. Assume the k components operate independently, and $P(c_j \text{ works }) = p_j$. What is the reliability of the system?

$$\begin{aligned} P(\text{system works}) &= P(\text{all components work}) \\ &= P(c_1 \text{ works} \cap c_2 \text{ works} \cap \cdots \cap c_k \text{ works}) \\ &= P(c_1 \text{ works}) P(c_2 \text{ work}) \cdots P(c_k \text{ works}) \\ &\overset{\text{Indep}}{=} \prod_{j=1}^{k} P(c_j \text{ works}) \\ &= \prod_{j=1}^{k} p_j \end{aligned}$$

Reliability Example

Example 6: Suppose a base is guarded by 3 radars (R_1, R_2, R_3), and the radars are independent of each other. The detection probability are ...

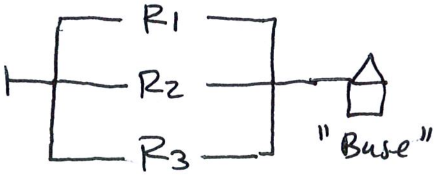
- $P(R_1) = P(R_1 \text{ detects}) = 0.95$
- $P(R_2) = P(R_2 \text{ detects}) = 0.98$
- $P(R_3) = P(R_3 \text{ detects}) = 0.99$

Does a system in *parallel* or *series* have higher reliability for this scenario?

- $P(\bar{R}_1) = P(R_1 \text{ doesn't work}) = 1 - P(R_1) = 1 - 0.95 = 0.05$
- $P(\bar{R}_2) = P(R_2$ " " $) = 1 - P(R_2) = 1 - 0.98 = 0.02$
- $P(\bar{R}_3) = P(R_3$ " " $) = 1 - P(R_3) = 1 - 0.99 = 0.01$

Reliability Example

Parallel



Reliability = Probability that the system works

$$P(\text{sys. works}) = P(\text{at least one radar works})$$
$$= 1 - P(\text{none works})$$
$$= 1 - P(\bar{R}_1 \cap \bar{R}_2 \cap \bar{R}_3)$$

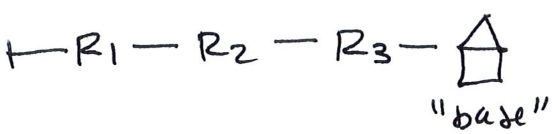
"and"

Indep

$$= 1 - [P(\bar{R}_1)P(\bar{R}_2)P(\bar{R}_3)]$$
$$= 1 - [(0.05)(0.02)(0.01)]$$
$$= 0.99999$$

Reliability Example

Series



$$P(\text{sys. works}) = P(\text{all radars work})$$
$$= P(R_1 \cap R_2 \cap R_3)$$

Indep

$$= P(R_1)P(R_2)P(R_3)$$
$$= (0.95)(0.98)(0.99)$$
$$= 0.922$$

[Parallel system has higher reliability than series system]