Lecture 4

Law of Total Probability & Bayes' Rule

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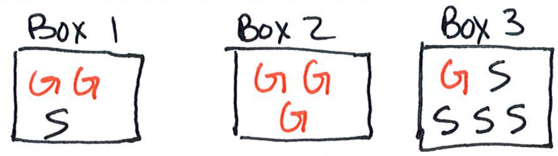
Tree Diagram

Tree Diagram

Example 1: Suppose you randomly select one of 3 boxes, and then randomly select a coin from inside the box. The contents of the boxes are ...



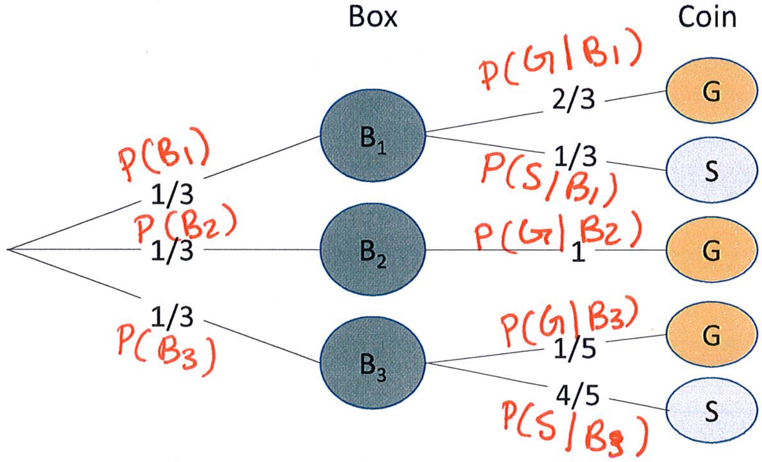
- Box 1: 2 gold coins, 1 silver coin
- Box 2: 3 gold coins
- Box 3: 1 gold coin, 4 silver coins

Let events $B_i = i^{th}$ box is selected for $i = 1, 2, 3$,
$G =$ gold coin selected, and $S =$ silver coin selected.

We can visualize this *step-wise procedure* with a *tree diagram*.

Using a Tree Diagram

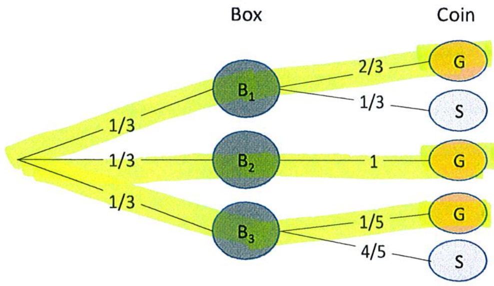
A tree diagram shows all possible outcomes of step-wise procedures



$P(B_i) = \frac{1}{3}$ for $i = 1, 2, 3$
$P(G|B_1) = \frac{2}{3}$, $P(S|B_1) = \frac{1}{3}$
$P(G|B_2) = 1$
$P(G|B_3) = \frac{1}{5}$, $P(S|B_3) = \frac{4}{5}$

Using a Tree Diagram Cont.

What is the probability of choosing a gold coin $P(G)$?



- What are the *"total"* different paths to get to gold coin? $P(A|B) = \frac{P(A \cap B)}{P(B)}$

  $(B_1 \cap G)$ or $(B_2 \cap G)$ or $(B_3 \cap G)$ Definition of conditional Probability
- These are disjoint events

$$P(G) = P(B_1 \cap G) + P(B_2 \cap G) + P(B_3 \cap G)$$
$$= P(B_1)P(G|B_1) + P(B_2)P(G|B_2) + P(B_2)P(G|B_2)$$
$$= \left(\tfrac{1}{3}\right)\left(\tfrac{2}{3}\right) + \left(\tfrac{1}{3}\right)(1) + \left(\tfrac{1}{3}\right)\left(\tfrac{1}{5}\right) = 0.62$$

This calculation is done using *Law of Total Probability.*

Law of Total Probability

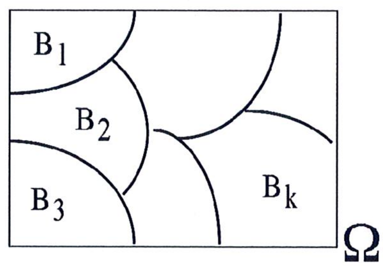
Cover/Partition

Definition:

A collection of events $B_1, \ldots B_k$ is a *cover* or *partition* of Ω if

1. the events are pairwise disjoint ($B_i \cap B_j = \emptyset$ for $i \neq j$), and
2. the union of the events is Ω ($\bigcup_{i=1}^{k} B_i = \Omega$).

We can represent a cover using a Venn diagram:



Note: In a tree diagram, the branches of the tree form a cover.

Law of Total Probability

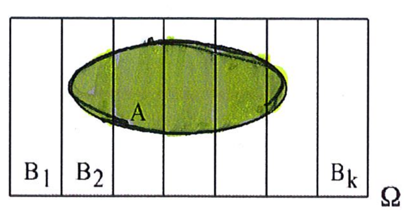
Theorem (Law of Total Probability)

If the collection of events $B_1, \ldots, B_k$ is a cover of Ω, and A is an event, then

$$P(A) = \sum_{i=1}^{k} \overbrace{P(A|B_i)P(B_i)}^{P(A\cap B_i)}.$$

<u>Proof</u>

- $A = (B_1 \cap A) \cup \ldots \cup (B_k \cap A)$
- $P(A) = P(B_1 \cap A) + \ldots + P(B_k \cap A)$
  $= P(A|B_1)P(B_1) + \ldots + P(A|B_k)P(B_k)$



Bayes' Rule

Bayes' Rule

Theorem (Bayes' Rule)

If $B_1, \ldots, B_k$ is a cover or partition of Ω, and A is an event, then

$$P(B_j|A) = \frac{P(A|B_j)P(B_j)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}$$

$\leftarrow P(A \cap B_j)$

$\leftarrow P(A)$

Why?

$$P(B_j|A) = \frac{P(A \cap B_j)}{P(A)} = \frac{P(A|B_j)P(B_j)}{\sum_{j=1}^{k} P(A|B_j)P(B_j)}$$

- Bayes rule $\rightarrow$ way to "flip" conditional probabilities.
- If we know $P(A|B_j)$ and $P(B_j)$, then we can obtain $P(B_j|A)$
- Extremely useful for real world applications!

Applying Bayes Rule

$P(N)=0.2$, $P(I) = 0.1$, $P(S)=0.7$

Example 2:

My email is divided into 3 folders: Normal, Important, Spam.
From past experience, the probability of emails belonging to these folders is 0.2, 0.1, and 0.7 respectively.

- Out of normal emails, the word "free" occurs with probability 0.01. $P(F|N) = 0.01$
- Out of important emails, "free" occurs with probability 0.01. $P(F|I)=0.01$
- Out of spam emails, "free" occurs with probability 0.9. $P(F|S)=0.9$

My spam filter reads an email that contains the word "free". What is the probability that this email is spam?

$P(S|F) = ?$

Applying Bayes Rule Cont.

Define events:

N = email is normal, I = email is important, S = email is spam
F = email contains "free", $\overline{F}$ = email doesn't contain "free"

Given:

$P(N) = 0.2$, $P(I) = 0.1$, $P(S) = 0.7$
$P(F|N) = 0.01$
$P(F|I) = 0.01$
$P(F|S) = 0.9$

$P(S|F) =?$ (This is what we want to know)

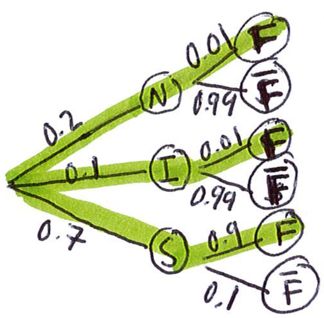
Applying Bayes Rule Cont.

What is the probability that my email is spam given that it contains the word "free"?

$$P(S|F) = \frac{P(S \cap F)}{P(F)}$$

$$= \frac{P(S)P(F|S)}{P(S)P(F|S) + P(I)P(F|I) + P(N)P(F|N)}$$

$= P(F)$ b/c of LOTP



$$= \frac{(0.7)(0.9)}{(0.7)(0.9) + (0.1)(0.01) + (0.2)(0.01)}$$

$$= \frac{0.63}{0.63 + 0.001 + 0.002}$$

$$= 0.995$$

multiply along branch & add up highlighted branches to get P(F) ← denominator

Applying Bayes Rule Cont.

Conceptual understanding

- Before knowing anything
  → probability that email is spam was $P(S) = 0.7$.
- After knowing that the email contains the word "free"
  → update probability based on this knowledge.
- After knowing the email contains "free"
  → probability of the email being spam is $P(S|F) = 0.995$.
- Since this probability is more than 50%, we can *classify* this email as spam.
- In machine learning/statistics, this procedure is called a *naive Bayes classifier*.

Example

Bayes' and LOTP Example

$P(C)$

C = cancer

$\bar{C}$ = no cancer

$+$ = tests pos.

$-$ = tests neg.

Example 3: Approximately 1% of women aged 40-50 have breast cancer. A woman with breast cancer has 90% chance of testing positive for cancer from a mammogram. A woman without breast cancer has a 5% chance of testing positive for cancer (called a "false positive"). What is the probability that a woman has breast cancer given that she tested positive?

$P(+|C)$

$P(+|\bar{C})$

Want to know $P(C|+) = ?$

Given

$P(C) = 0.01$ $\quad$ $P(\bar{C}) = 0.99$

$P(+|C) = 0.90$ $\quad$ $P(-|C) = 0.10$

$P(+|\bar{C}) = 0.05$ $\quad$ $P(-|\bar{C}) = 0.95$

What is $P(C|+) = ?$
Since we want to "flip" the condition, use Bayes' Rule

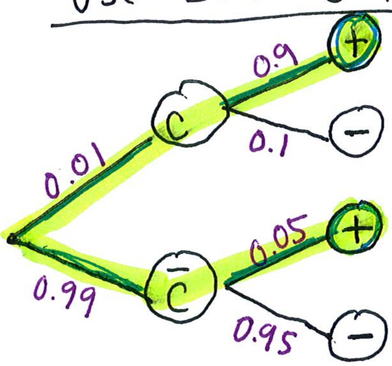
Bayes' and LOTP Example Cont.

Bayes

$$P(C|+) = \frac{P(C)P(+|C)}{P(+)} = \frac{P(C)P(+|C)}{P(C)P(+|C) + P(\bar{C})P(+|\bar{C})}$$

using LOTP

Use Law of Total Probability to get Denominator



↳ obtained from branches of tree diagram

$$P(+) = P(C)P(+|C) + P(\bar{C})P(+|\bar{C})$$
$$= (0.01)(0.9) + (0.99)(0.05)$$
$$= 0.009 + 0.0495$$
$$= 0.0585$$

Bayes' and LOTP Example Cont.

Back to Bayes' Rule

$$P(C|+) = \frac{P(C)P(+|C)}{P(+)}$$
$$= \frac{P(C)P(+|C)}{P(C)P(+|C) + P(\bar{C})P(+|\bar{C})}$$
$$= \frac{(0.01)(0.90)}{(0.01)(0.90) + (0.99)(0.05)}$$
$$= 0.1538$$

End Exam 1 Material