Lecture 5

Random Variables & Distributions

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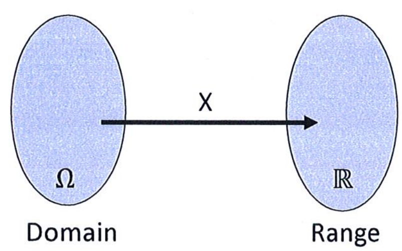
Random Variable

Types of Random Variables

Types of Random Variables



Two types of random variables:

Discrete Random Variable	Continuous Random Variable
Sample space (Ω) maps to finite or countably infinite set in $\Re$	Sample space (Ω) maps to an uncountable set in $\Re$.
Ex: $\{1, 2, 3\}$, $\{1, 2, 3, 4, \ldots\}$	Ex: $(0, \infty)$, $(10, 20)$

Image of a Random Variable

Definition

The *image* of a random variable is defined as the values the random variable can take on.

$$Im(X) = \{x : x = X(\omega) \text{ for some } \omega \in \Omega\}$$

whatever the realized values of your R.V can be

Example 2:

1. Put a disk drive into service. Let Y = time till the first major failure. $Im(Y) = (0, \infty)$. — continuous
   Image of Y is an interval (uncountable)
   $\rightarrow Y$ is a continuous random variable.
2. Flip a coin 3 times. Let X = # of heads obtained.
   $Im(X) = \{0, 1, 2, 3\}$. Image of X is a finite set
   $\rightarrow X$ is a discrete random variable.

Probability Mass Function

Probability Mass Function

Two things to know about a random variable X:

(1) What are the values X can take on? (what is its image?)

(2) What is the probability that X takes on each value?

(1) and (2) together gives the *probability distribution* of X.

Breakdown of X and its corresponding probabilities

Definition

Let X be a discrete random variable.

The *probability mass function (pmf)* of X is $p_X(x) = P(X = x)$.

little x (realized value)

capital X (represents my R.V)

Properties of pmf:

probabilities must be b/w 0 & 1 → 1. $0 \le p_X(x) \le 1$

2. $\sum_x p_X(x) = 1$ ← probabilities must sum to 1

Probability Mass Function Cont.

Example 3: Which of the following are *valid* probability mass functions (pmfs)?

✓ 1.

x	-3	-1	0	5	7
$p_X(x)$	0.1	0.45	0.15	0.25	0.05

↔ Can be written as function

$$P_X(x) = \begin{cases} 0.1 & \text{for } x=-3 \\ 0.45 & \text{for } x=-1 \\ 0.15 & \text{for } x=0 \\ 0.25 & \text{for } x=5 \\ 0.05 & \text{for } x=7 \end{cases}$$

NOT Valid ✗ 2.

y	-1	0	1.5	3	4.5
$p_Y(y)$	0.1	0.45	0.25	-0.05	0.25

$P_Y(3) = -0.05$ probabilities must be b/w 0 & 1

NOT Valid ✗ 3.

z	0	1	3	5	7
$p_Z(z)$	0.22	0.18	0.24	0.17	0.18

probabilities should sum to 1

$\sum_z P_Z(z) = 0.99 \neq 1$

Probability Mass Function Cont.

Example 4: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.

1. Define the random variable X.

   X = # of heads obtained in 3 coin flips

2. What is the image of X?

   $Im(X) = \{0, 1, 2, 3\}$

3. What is the pmf of X? (find $p_X(x)$ for all x)

$$P(X=0) = P(TTT) = (\tfrac{1}{2})(\tfrac{1}{2})(\tfrac{1}{2}) = (\tfrac{1}{2})^3 = \tfrac{1}{8}$$

$$P(X=1) = P(HTT) + P(THT) + P(TTH) = 3(\tfrac{1}{2})^3 = \tfrac{3}{8}$$

$$P(X=2) = P(HHT) + P(HTH) + P(THH) = 3(\tfrac{1}{2})^3 = \tfrac{3}{8}$$

$$P(X=3) = P(HHH) = (\tfrac{1}{2})^3 = \tfrac{1}{8}$$

Probability Mass Function Cont.

We can write PMF as a table

x	0	1	2	3
$P_X(x)$	1/8	3/8	3/8	1/8

or write PMF as a function

$$P_X(x) = \begin{cases} \frac{1}{8} & \text{for } x = 0, 3 \\ \frac{3}{8} & \text{for } x = 1, 2 \end{cases}$$

- What is prob of obtaining 2 heads?

  $P(X=2) = 3/8$

- What is prob of obtaining 2 or 3 heads?

  $P(\{X=2\} \cup \{X=3\}) = P(X=2) + P(X=3)$

  $= 3/8 + 1/8 = 1/2$

Cumulative Distribution Function

Cumulative Distribution Function

Definition

The *cumulative distribution function (cdf)* of X is

$$F_X(t) = P(X \leq t)$$

- The pmf is $P_X(x) = P(X = x)$, the probability that R.V. X is equal to value x.
- The cdf is $F_X(t) = P(X \leq t)$, the probability that R.V. X is less than or equal to t.

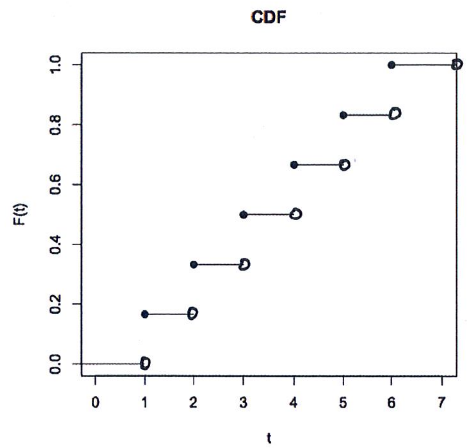
Relationship between pmf and cdf

- $F_X(t) = P(X \leq t) = \sum_{x \leq t} p_X(x) = \sum_{x \leq t} P(X = x)$

Properties of CDFs

Properties of CDFs

1. $0 \leq F_X(t) \leq 1$
2. F_X is non-decreasing (if $a \leq b$, then $F(a) \leq F(b)$.
3. $\lim_{t \to -\infty} F_X(t) = 0$ and $\lim_{t \to \infty} F_X(t) = 1$
4. F_X is right-continuous with respect to t



CDF's for discreter R.Vs are step functions

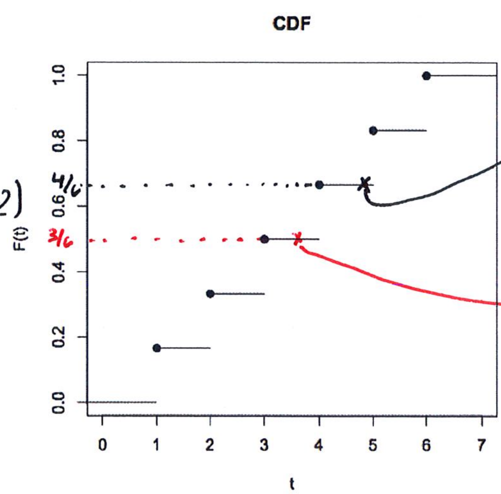
Cumulative Distribution Function Cont.

Example 6: Roll a fair die. Let X = the number of dots on face up

x	1	2	3	4	5	6
(pmf) $p_X(x)$	1/6	1/6	1/6	1/6	1/6	1/6
(cdf) $F_X(x)$	1/6	2/6	3/6	4/6	5/6	1

$F_X(1) = P(X \leq 1)$
$= P(X=1) = 1/6$

$F_X(2) = P(X \leq 2)$
$= P(X=1) + P(X=2)$
$= \frac{1}{6} + \frac{1}{6}$
$= \frac{2}{6}$

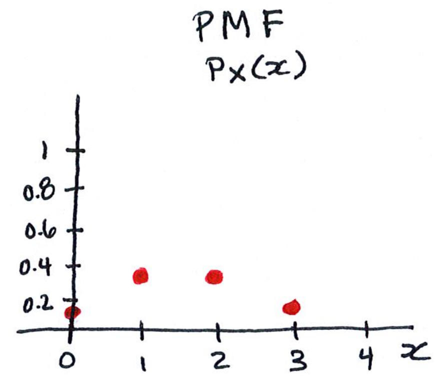


$F_X(4.999\bar{9}) = P(X \leq 4.999\bar{9})$
$= P(X=1) + P(X=2)$
$+ P(X=3) + P(X=4)$
$= \frac{4}{6}$

$F_X(3.5) = P(X \leq 3.5)$
$= P(X=1) + P(X=2)$
$+ P(X=3)$
$= 3/6$

Cumulative Distribution Function Cont.

Example 7: Suppose you toss 3 coins, and observe the face up for each flip. We are interested in the number of heads we obtain in 3 coin tosses.



From example 4, the pmf is

x	0	1	2	3
(pmf) $p_X(x)$	1/8	3/8	3/8	1/8
(cdf) $F_X(x)$	1/8	4/8	7/8	1

What is the cdf of X?