Lecture 6

Expected Value and Variance

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Expected Value

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Example 1: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

What number of heads do we "expect" to get?

0 obtained $\frac{1}{8}$ of the time

1 obtained $\frac{3}{8}$ of the time

2 obtained $\frac{3}{8}$ of the time

3 obtained $\frac{1}{8}$ of the time

Intuitively, we can think about taking $0\left(\frac{1}{8}\right) + 1\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 3\left(\frac{1}{8}\right)$ as the "expected" number of heads

Expected Value

Definition

Let X be a discrete random variable. The *expected value* or *expectation* of $h(X)$ is

$$E\left[h(X)\right] = \sum_x h(x)p_X(x) = \sum_x h(x)P(X = x)$$

- The **MOST IMPORTANT** version of this is when h(x) = x

$$E(X) = \sum_x xp_X(x) = \sum_x xP(X = x)$$

- $E(X)$ is usually denoted by μ
- $E(X)$ is the weighted average of the x's, where the weights are the probabilities of the x's.

Expected Value Cont.

Example 2: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the expected value of X.

$$\begin{aligned} E(X) &= \sum_x x p_X(x) \\ &= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3) \\ &= \end{aligned}$$

Variance

Variance & Standard Deviation

Definition

The *variance* of a random variable X is

$$Var(X) = E\left[(X - E(X))^2\right] = \sum_x (x - E(X))^2 \cdot p_X(x)$$

The *standard deviation* of a random variable X is

$$\sigma = \sqrt{Var(X)}$$

- $Var(X)$ is usually denoted by σ^2
- Units for variance is squared units of X.
- Units for the standard deviation is same as units for X.

SHORT CUT (use this formula to find variance)

$$\begin{aligned} Var(X) &= E(X^2) - \left[E(X)\right]^2 \\ &= \sum_x x^2 P(X = x) - \left[\sum_x xP(X = x)\right]^2 \end{aligned}$$

Variance Cont.

Example 3: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

- $E(X) = \sum_x x p_X(x) =$

- $E(X^2) = \sum_x x^2 p_X(x) =$

Variance Cont.

- $Var(X) = E(X^2) - \left[E(X)\right]^2 =$

- $\sigma = \sqrt{Var(X)} =$

Properties of E(X) & Var(X)

Operations with E(X) and Var(X)

X, Y are random variables; a, b, c are constants.

Operations with $E(\cdot)$

1. $E(aX) = aE(X)$
2. $E(aX + bY + c) = aE(X) + bE(Y) + c$

Operations with $Var(\cdot)$

3. $Var(aX) = a^2 Var(X)$
4. $Var(aX + b) = a^2 Var(X)$
5. $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$
   (when X,Y are independent, Cov(X,Y) $=$ 0. We'll discuss more about independence and define covariance in Lecture 9)

Chebyshev's Inequality

Chebyshev's Inequality

For any positive real number k, and random variable X with variance σ^2:

$$P(|X - E(X)| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

- bounds the probability that X lies within a certain number of standard deviations from $E(X)$