Exam 1 — In class — Fri, 9/27
- Lectures 1-4 (upto & including Bayes' Rule)
- Closed book
- Bring 1 page (front & back) note sheet
- Bring calculator

Lecture 6

Expected Value and Variance

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Expected Value (Expectation)

Expected Value

Example 1: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

What number of heads do we "expect" to get?

0 obtained $\frac{1}{8}$ of the time

1 obtained $\frac{3}{8}$ of the time

2 obtained $\frac{3}{8}$ of the time

3 obtained $\frac{1}{8}$ of the time

Intuitively, we can think about taking $0(\frac{1}{8}) + 1(\frac{3}{8}) + 2(\frac{3}{8}) + 3(\frac{1}{8})$ as the "expected" number of heads

Expected Value

Definition

Let X be a discrete random variable. The *expected value* or *expectation* of $h(X)$ is — a function of X

$$E\left[h(X)\right] = \sum_x h(x)p_X(x) = \sum_x h(x)P(X = x)$$

identity function

- The **MOST IMPORTANT** version of this is when h(x) = x

$$E(X) = \sum_x x p_X(x) = \sum_x xP(X = x)$$

- $E(X)$ is usually denoted by μ = "mu"
- $E(X)$ is the weighted average of the x's, where the weights are the probabilities of the x's.

Expected Value Cont.

Example 2: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the expected value of X.

$$E(X) = \sum_x x p_X(x)$$
$$= 0P(X=0) + 1P(X=1) + 2P(X=2) + 3P(X=3)$$
$$= (0)(\tfrac{1}{8}) + (1)(\tfrac{3}{8}) + (2)(\tfrac{3}{8}) + (3)(\tfrac{1}{8})$$
$$= 1.5$$

Variance

Variance & Standard Deviation

Definition Gives us idea of how spread apart the x's are wrt. E(X)

The *variance* of a random variable X is

$$Var(X) = E\left[(X - E(X))^2\right] = \sum_x (x - E(X))^2 \cdot p_X(x)$$

The *standard deviation* of a random variable X is

"sigma"

$$\sigma = \sqrt{Var(X)}$$

- $Var(X)$ is usually denoted by σ^2
- Units for variance is squared units of X.
- Units for the standard deviation is same as units for X.

SHORT CUT (use this formula to find variance)

Usually easier to find variance using this

$$Var(X) = E(X^2) - [E(X)]^2$$

$$= \sum_x x^2 P(X = x) - \left[\sum_x x P(X = x)\right]^2$$

Variance Cont.

Example 3: Flip a coin 3 times. Let $X = \#$ of heads obtained in 3 flips. The probability mass function (pmf) of X is

x	0	1	2	3
$p_X(x)$	1/8	3/8	3/8	1/8

Calculate the variance and standard deviation of X.

Use the short cut to calculate Var(X)

$Var(X) = E(X^2) - [E(X)]^2$

- $E(X) = \sum_x x p_X(x) = 1.5$ (example 2)
- $E(X^2) = \sum_x x^2 p_X(x) =$
  $= (0^2)(\frac{1}{8}) + (1^2)(\frac{3}{8}) + (2^2)(\frac{3}{8}) + (3^2)(\frac{1}{8})$
  $= 3$

Variance Cont.

σ^2

- $Var(X) = E(X^2) - [E(X)]^2 = 3 - (1.5)^2 = 0.75$

don't forget to square

- $\sigma = \sqrt{Var(X)} = \sqrt{0.75} = 0.866$

standard deviation

Properties of E(X) & Var(X)

Operations with E(X) and Var(X)

X, Y are random variables; a, b, c are constants.

Operations with $E(\cdot)$

$E(X) = 6$

$E(4X) = 4E(X) = 4(6)$

1. $E(aX) = aE(X)$
2. $E(aX + bY + c) = aE(X) + bE(Y) + c$

Operations with $Var(\cdot)$

3. $Var(aX) = a^2 Var(X)$
4. $Var(aX + b) = a^2 Var(X)$

- square constants when you pull it out of Variance
- ignore any constants that are not attached to a R.V

5. $Var(aX + bY) = a^2 Var(X) + b^2 Var(Y) + 2ab\underline{Cov(X, Y)}$
   (when X,Y are independent, Cov(X,Y) = 0. We'll discuss more about independence and define covariance in Lecture 9)

$Var(X) = 10$

$Var(4X) = 4^2 Var(X) = (4^2)(10)$

Chebyshev's Inequality

Chebyshev's Inequality

For any positive real number k, and random variable X with variance σ^2:

$$P(|X - E(X)| \leq k\sigma) \geq 1 - \frac{1}{k^2}$$

- bounds the probability that X lies within a certain number of standard deviations from $E(X)$