# Lecture 8

Discrete Distributions: Geometric and Poisson Distributions

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# Geometric Distribution

## Geometric Distribution

Set up:

- Experiment where each trial is Bernoulli (only 2 outcomes) with P(success) = $p$
- Repeat the trials until you obtain the *first success*.

$\underline{\text{F}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ $\underline{\text{F}}$ $\cdots$ $\underline{\text{F}}$ $\underline{\text{S}}$

$$X = \text{“ \# of trials until first success”}$$ (including 1st success)

This random variable $X$ follows a *Geometric Distribution*

is distributed

$$X \sim Geo(p)$$ parameter

where $p$ is the probability of success for each trial. p = P(success) for each trial



## Geometric Random Variable

- Probability Mass Function (pmf)
  1. $Im(X) = \{1, 2, 3, 4, \ldots, \} = \mathbb{N}$
  2. $P(X = x) = (1-p)^{x-1}p$

$\underbrace{\underline{\text{F}}\ \underline{\text{F}}\ \underline{\text{F}} \ldots \underline{\text{F}}}_{x-1 \text{ failures w/ probability } 1-p}\ \underbrace{\underline{\text{S}}}_{1 \text{ success w/ prob. } p}$

$$p_X(x) = \underbrace{(1-p)^{x-1}}_{\text{failures}}\underbrace{p}_{\text{success}} \quad \text{for } x = 1, 2, 3, \ldots$$

- Cumulative distribution function (cdf)

$$F_X(t) = P(X \le t) = 1 - (1-p)^{\lfloor t \rfloor}$$

$\lfloor t \rfloor$ = "floor" of t

$\lfloor 3.6 \rfloor = 3$

$\lfloor 3.2 \rfloor = 3$

$\lfloor 5.\overline{999} \rfloor = 5$

Why?

- $P(X > t) = (1-p)^{\lfloor t \rfloor}$ because this is the probability that the first $\lfloor t \rfloor$ trials are failures
- $P(X \le t) = 1 - P(X > t) = 1 - (1-p)^{\lfloor t \rfloor}$

## Geometric Random Variable Cont.

- Expected Value

$$E(X) = \sum_{x=1}^{\infty} x(1-p)^x p = \cdots = \frac{p}{(1-[1-p])^2} = \frac{1}{p}$$

- Variance

$$Var(X) = \sum_{i=1}^{\infty} \left(i - \frac{1}{p}\right)^2 (1-p)^i p = \cdots = \frac{1-p}{p^2}$$

- "Memoryless" Property of Geometric Dist.

$$P(X \geq i + j | X \geq i) = P(X \geq j) \text{ for } i, j = 0, 1, 2, \ldots$$



## Geometric Distribution Examples

## Geometric Distribution Examples

Example 1: Suppose we have an unfair 2−sided coin with $p = P(Head) = 0.3$. We flip a coin until we get our first head, and stop flipping once we obtain the head.

$p = P(\text{success}) = P(\text{Heads}) = 0.3$

What is the probability that ...

1. the first head occurs on the third flip?
2. we get the first head before the third flip?
3. we have to flip the coin at least 3 times, but at most 7 times to get the first head?
4. What is the expected number of flips until we obtain the first head?
5. What is the variance?



## Geometric Distribution Examples

Start by defining the R.V and stating it's distribution.

$Y = \#$ of flips until $1^{st}$ success / head

$$Y \sim Geo(p) \equiv Geo(0.3)$$

$p = 0.3$

1. What is the probability that the first head occurs on the third flip?

$\underline{F}\ \underline{F}\ \underline{S}$

$$P(Y=3) = (1-0.3)^{3-1}(0.3) = (0.7)^2(0.3) = 0.147$$

## Geometric Distribution Examples

2. What is the probability that we get the first head before the third flip? $P(Y < 3) = ?$

using PMF: $P(Y<3) = P(Y \le 2) = P_Y(1) + P_Y(2)$

$= 0.3 + (1-0.3)^1(0.3)$

$= 0.3 + (0.7)(0.3)$

$= 0.51$

using CDF: $P(Y<3) = P(Y \le 2) = F_Y(2)$

$= 1-(1-0.3)^2$

$= 0.51$



## Geometric Distribution Examples

3. What is the probability that we have to flip the coin at least 3 times, but at most 7 times? $P(3 \le Y \le 7) = ?$

using PMF: $P(3 \le Y \le 7) = P_Y(3) + P_Y(4) + P_Y(5) + P_Y(6) + P_Y(7)$

$= \cdots$

$= 0.4076$

using CDF: $P(3 \le Y \le 7) = P(Y \le 7) - P(Y<3)$

$= P(Y \le 7) - P(Y \le 2)$

$= F_Y(7) - F_Y(2)$

$= [1-(1-0.3)^7] - [1-(1-0.3)^2]$

$= 0.9176 - 0.51 = 0.4076$

4. What is the expected value?

$E(Y) = 1/p = 1/0.3 = 3.33$

5. What is the variance?

$Var(Y) = \frac{1-p}{p^2} = \frac{1-0.3}{0.3^2} = 7.78$

# Poisson Distribution

## Poisson Distribution

Set up: The Poisson distribution is used to model the number of ("rare") events occurring in a fixed interval of time.

Examples of Poisson R.Vs

- $Y = \#$ of meteorites that strike Earth in a year
- $Z = \#$ of patients arriving to emergency room from 10-11 pm

Define the random variable

$$X = \text{"\# of events occurring during an interval"}$$

This random variable X follows a *Poisson Distribution*

"follows"

$$X \sim Pois(\lambda)$$

"lambda" $\lambda$

where $\lambda > 0$ is called the rate parameter

## Poisson R.V. Summary

- Probability Mass Function (pmf)

$$P(X=x) = p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} \quad \text{for } x = 0, 1, 2, 3, \ldots$$

where $\lambda > 0$ is the rate parameter.

- Cumulative Distribution Function (cdf)

$$F_X(t) = P(X \leq t) = \sum_{x=0}^{\lfloor t \rfloor} p_X(x)$$

- Expected Value: $E(X) = \lambda$

- Variance: $Var(X) = \lambda$



# Poisson Distribution Examples

## Poisson Distribution Examples

Example 2: Suppose the number of customers entering West Street Deli can be modeled using a Poisson distribution. Customers enter the deli at an average rate of 10 customers every ~~10~~ 15 minutes during the lunch rush.

rate interval

Between 12pm and 12:~~10~~15pm today, what is the probability that . . .

1. exactly 3 customers enter?
2. at most 3 customers enter?
3. at least 4 customers enter?
4. Between 8 and 10 customers enter? (inclusive)
5. What is the expected value of the random variable?
6. What is the variance of the random variable?



## Poisson Distribution Examples

Start by defining the R.V and stating it's distribution.

$X$ = # of customers entering W.S Deli b/w 12-12:~~10~~15pm

$X \sim \text{Pois}(\lambda) \equiv \text{Pois}(10)$

PMF

$$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!} = \frac{e^{-10}(10)^x}{x!}$$

1. What is the probability that exactly 3 customers enter?

$$P(X=3) = p_X(3) = \frac{e^{-10}(10)^3}{3!} = 0.00757$$

CDF

$$F_X(t) = P(X \leq t) = \sum_{x=0}^{\lfloor t \rfloor} p_X(x)$$

or use Poisson Table

## Poisson Distribution Examples

2. What is the probability that at most 3 customers enter

Using CDF $\quad P(X \leq 3) = F_X(3) = 0.0103$ (From Poisson Table) Appendix A

or

Using ~~PDF~~ PMF

$$P(X \leq 3) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$$

$$= \frac{e^{-10}(10)^0}{0!} + \frac{e^{-10}(10)^1}{1!} + \frac{e^{-10}(10)^2}{2!} + \frac{e^{-10}(10)^3}{3!}$$

$$= 0.0103$$



## Poisson Distribution Examples

3. What is the probability that at least 4 customers enter?

$$P(X \geq 4) = 1 - P(X < 4)$$

$$= 1 - P(X \leq 3)$$

$$= 1 - F_X(3)$$

use Appendix A Poisson Table

$$= 1 - 0.0103$$

$$= 0.9897$$

# Poisson Distribution Examples

4. What is the probability that between 8 and 10 customers enter (inclusive)

using ~~CDF~~ PMF

$$P(8 \le X \le 10) = P(X=8) + P(X=9) + P(X=10)$$
$$= \frac{e^{-10}(10)^8}{8!} + \frac{e^{-10}(10)^9}{9!} + \frac{e^{-10}(10)^{10}}{10!}$$
$$= 0.3628$$

Using CDF

$$P(8 \le X \le 10) = P(X \le 10) - P(X < 8)$$
$$= P(X \le 10) - P(X \le 7)$$
$$= F_X(10) - F_X(7)$$
$$= 0.5830 - 0.2202 = 0.3628$$

5. What is the expected value of the random variable?

$$E(X) = \lambda = 10$$

6. What is the variance of the random variable?

$$Var(X) = \lambda = 10$$