Lecture 9

Joint PMF

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Joint PMF

Joint Probability Mass Function

Motivation:

- Often, real problems deal with more than 1 variable
- Not sufficient to model the variables separately
- Need to consider their *joint* behavior

Definition

For two discrete variables X and Y, the *joint probability mass function (pmf)* is defined as:

$$p_{X,Y}(x,y) \equiv P(X=x, Y=y) = P(\{X=x\} \cap \{Y=y\})$$

Joint PMF Example

Example 1:

A box contains 5 unmarked processors of different speeds:

speed (mHz)	400	450	500
count	2	1	2

X = speed of the first selected processor
Y = speed of the second selected processor

The *(joint) probability table* below gives the probabilities for each processor combination:

		2nd processor (Y)		
	mHz	400	450	500
1st proc. (X)	400	0.1	0.1	0.2
	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

Joint PMF Example Cont.

1. What is the probability that $X = Y$?

		2nd processor (Y)		
	mHz	400	450	500
	400	0.1	0.1	0.2
1st proc. (X)	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

$$
\begin{aligned}
&P(X = Y) \\
&= p_{X,Y}(400, 400) + p_{X,Y}(450, 450) + p_{X,Y}(500, 500) \\
&= 0.1 + 0 + 0.1 \\
&= 0.2
\end{aligned}
$$

Joint PMF Example Cont.

2. What is the probability that $X > Y$?

		2nd processor (Y)		
	mHz	400	450	500
	400	0.1	0.1	0.2
1st proc. (X)	450	0.1	0.0	0.1
	500	0.2	0.1	0.1

X

In other words, what is the probability that 1st processor has higher speed than 2nd processor?

Y

$$
\begin{aligned}
&P(X > Y) \\
&= p_{X,Y}(450, 400) + p_{X,Y}(500, 400) + p_{X,Y}(500, 450) \\
&= 0.1 + 0.2 + 0.1 \\
&= 0.4
\end{aligned}
$$

Marginal PMF

Marginal Probability Mass Function

We obtain the *marginal pmf* from the *margins* of the probability table.

This is obtained by summing up the cells row-wise or column-wise.

Definition

The *marginal probability mass functions* $p_X(x)$ and $p_Y(y)$ can be obtained from the joint pmf $p_{X,Y}(x,y)$ by

$$\left.\begin{aligned} p_X(x) &= \sum_y p_{X,Y}(x,y) \\ p_Y(y) &= \sum_x p_{X,Y}(x,y) \end{aligned}\right\} \text{Result of Law of Total Probability}$$

Marginal PMF Cont.

		2nd processor (Y)			
	mHz	400	450	500	$p_X(x)$
1st proc. (X)	400	0.1	0.1	0.2	0.4
	450	0.1	0.0	0.1	0.2
	500	0.2	0.1	0.1	0.4
	$p_Y(y)$	0.4	0.2	0.4	1

marginal PMF of X

marginal PMF of Y

Thus, the marginal pmf are . . .

x	400	450	500
$p_X(x)$	0.4	0.2	0.4

y	400	450	500
$p_Y(y)$	0.4	0.2	0.4

Expectation

Expected Value

Definition

The *expected value* of a function of several variables is

$$E[h(X, Y)] \equiv \sum_{x,y} h(x, y) p_{X,Y}(x, y)$$

- The **MOST IMPORTANT** application of this will be for calculating covariance (next slide).
- To calculate the covariance, we will need $E(XY)$.

  Take $h(X, Y) = X \cdot Y$, and plug in into expected value formula

$$E(XY) = \sum_{x,y} xy p_{X,Y}(x, y)$$

Covariance

Covariance

For two variables, we can measure how "similar" their values are using *covariance* and *correlation*.

Definition

The *covariance* of 2 random variables X, Y is given by

$$Cov(X, Y) = E\big[(X - E(X))(Y - E(Y))\big]$$

- This definition is similar to Var(X).
- In fact, $Cov(X, X) = Var(X)$
- In practice, use SHORT CUT formula to obtain covariance:

Recall

$Var(X) = E(X^2) - E(X)^2$

$$Cov(X, Y) = E(XY) - E(X)E(Y)$$

Correlation

Correlation

Definition

The *correlation* between 2 random variables X, Y is given by

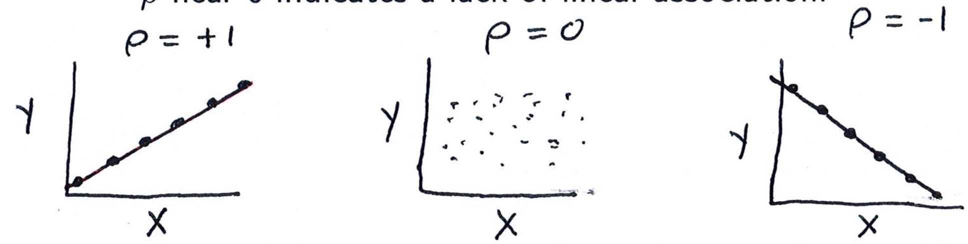
$$\rho = Corr(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X) \cdot Var(Y)}}$$

Properties of Correlation (ρ):

"rho" = ρ

- ρ is a measure of linear association between X and Y.
- $-1 \le \rho \le 1$
- ρ near ± 1 indicates a strong linear relationship

  ρ near 0 indicates a lack of linear association.



Correlation Example

Back to Example 1:

3. What is the correlation between X and Y? $Corr(X,Y) = \rho_{XY}$

In this example,

$E(X) = E(Y) = 450$

$Var(X) = Var(Y) = 2000.$

① Find $Cov(X,Y)$

$$Cov(X,Y) = \underbrace{E(XY)}_{?} - \underbrace{E(X)}_{450}\underbrace{E(Y)}_{450}$$

$$E(XY) = \sum_{x,y} xy\, P_{X,Y}(x,y)$$

$$= (400)(400)(0.1) + (400)(450)(0.1) + \ldots + (500)(500)(0.1)$$

$$= 202{,}000$$

$$Cov(X,Y) = E(X,Y) - E(X)E(Y)$$

$$\Rightarrow = 202000 - (450)(450) = -500$$

Correlation Example Cont.

② Calculate Correlation

$$\rho = \text{Corr}(X,Y) = \frac{\text{Cov}(X,Y)}{\sqrt{\text{Var}(X)\text{Var}(Y)}}$$

$$= \frac{-500}{\sqrt{(2000)(2000)}}$$

$$= -0.25$$

(indicates a weak, negative, linear relationship b/w X & Y)

Independence

Independence

Recall that random variables X, Y are *independent* if all events of the form $\{X = x\}$ and $\{Y = y\}$ are independent.

For independence, we need

$$\underbrace{p_{X,Y}(x, y)}_{\text{joint}} = \overbrace{p_X(x) p_Y(y)}^{\text{marginal}} \text{ for all } x, y$$

- check if the above holds for all possible combos of x and y
- If we find one contradiction, then we do not have independence

Find $Cov(X,Y)$ If $Cov(X,Y) \neq 0$ Then X, Y are not independent

SHORT CUT: If two random variables are independent, then they have $Cov(X, Y) = 0$.

Note: The converse is not always true

- All independent random variables have 0 covariance
- Some dependent random variables also have 0 covariance

Independence Example

Back to Example 1:

4. Are X and Y independent?

- Check whether $p_{X,Y}(x, y) = p_X(x)p_Y(y)$ for x, y pairs.
- $p_{X,Y}(450, 450) = 0 \neq (0.2)(0.2) = p_X(450)p_X(450)$
- X and Y are NOT independent.

Alternatively ...

- $Cov(X, Y) = -500 \neq 0$
- X and Y are NOT independent.

More on Expectation and Variance

More on Variance

Definition

Let X and Y be random variables, and a,b,c be real numbers. constants

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y) + 2abCov(X, Y)$$

- Recall that for independent random variables, $Cov(X, Y) = 0$
- Thus if X and Y are independent, this simplifies to

$$Var(aX + bY + c) = a^2 Var(X) + b^2 Var(Y)$$

More on Expected Value

Definition

Let X and Y be random variables.

$$E(XY) = \sum_{x,y} xyp_{X,Y}(x,y)$$

- If X and Y are independent, this simplifies to

$$\begin{aligned} E(XY) &= \sum_{x,y} xyp_X(x)p_Y(y) \\ &= \sum_x xp_X(x) \sum_y yp_Y(y) \\ &= E(X)E(Y) \end{aligned}$$

- If X and Y are independent, E(XY) = E(X)E(Y)

Additional Example

Y \ X	6	10	12	14	
1	0	0.1	0.1	0.3	0.5
45	0.1	0	0.2	~~0.3~~ 0.2	0.5
	0.1	0.1	0.3	0.5	

Find marginal of X and of Y

x	6	10	12	14
$P_X(x)$	0.1	0.1	0.3	0.5

y	1	45
$P_Y(y)$	0.5	0.5

$$E(XY) = \sum_{x,y} xy\, P_{X,Y}(x,y)$$
$$= (6)(1)(0) + (10)(1)(0.1) + (12)(1)(0.1) + \dots$$
$$\dots (14)(45)(0.2) =$$

$$E(Y) = \sum_y y P_Y(y) = (1)(0.5) + (45)(0.5) =$$